

Scalar products in models with $GL(3)$ trigonometric R -matrix. General case

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Abstract

We study quantum integrable models with $GL(3)$ trigonometric R -matrix solvable by the nested algebraic Bethe ansatz. We analyze scalar products of generic Bethe vectors and obtain an explicit representation for them in terms of a sum with respect to partitions of Bethe parameters. This representation generalizes known formula for the scalar products in the models with $GL(3)$ -invariant R -matrix.

1 Introduction

The algebraic Bethe ansatz [1, 2, 4, 3] allows one to obtain the spectrum of quantum Hamiltonians in many models of physical interest. Calculation of correlation functions also can be performed in the framework of this method. The last problem in many cases can be reduced to the calculation of scalar products of Bethe vectors. In our previous work [5] we began a systematic study of scalar products in quantum integrable models with $GL(3)$ trigonometric R -matrix. In that paper we have calculated the highest coefficients of the scalar product and

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found their properties. Using these results we continue the investigation of the scalar products in the present paper.

For the $GL(2)$ -based models the study of scalar products of Bethe vectors in the framework of the algebraic Bethe ansatz was initiated in the works [6, 7, 8, 9]. There it was used for the calculation of the correlation functions in the models of one-dimensional bosons and XXZ Heisenberg chain. In the work [10] a determinant representation for the scalar product of an arbitrary Bethe vector and an eigenvector of the transfer matrix was obtained. This result allowed later to obtain multiple integral representations for correlation functions in various quantum integrable models [11, 12, 13, 14, 15, 16]. The determinant representation for the scalar product also was used for the calculation of form factors of local operators [17, 18, 19]. These results were used for the analytical [20, 21] and numerical analysis of correlation functions [22, 23, 24, 25].

Formally, the scalar product of Bethe vectors should play the same role in the models described by the higher rank algebras. However this case is much more sophisticated from the technical viewpoint. Up to now the only representation for the scalar product for the models with $GL(3)$ -invariant R -matrix is known due to N. Reshetikhin [26]. This representation was given in terms of a sum with respect to partitions of Bethe parameters (so-called sum formula). Such type of formulas are not appropriate neither for analytical nor for numerical calculations. However, they can be used to produce more compact formulas, in particular, to find determinant representation for some particular cases of the scalar products. Representations of this type were obtained in [27, 28, 29] for some important particular cases of the scalar products in quantum integrable models with $GL(3)$ -invariant R -matrix.

The question arises about the generalization of the Reshetikhin representation to the trigonometric case. In distinction of the $GL(2)$ -based models this generalization is not straightforward. In particular, in the paper [5] we have shown that in the models with $GL(3)$ trigonometric R -matrix there exist two highest coefficients, while in the case of $GL(3)$ -invariant R -matrix there is only one highest coefficient.

The main goal of this paper is to derive such the generalization. In the paper [5] we have made the first step on this way. Now we go further and obtain an analog of the Reshetikhin formula for the scalar product for the models with $GL(3)$ trigonometric R -matrix. We use the same method as in [5]. It is based on the explicit representations for the dual Bethe vectors and formulas of multiple action of the monodromy matrix entries onto Bethe vectors. This method is straightforward, although it is a bit technical. In many respects it relies on the properties of the highest coefficients established in [5].

The content of the paper is as follows. In section 2 we describe the model under consideration and introduce necessary notations. In section 3 we present the main result of the paper: the sum formula for the scalar product of Bethe vectors in the models with $GL(3)$ trigonometric R -matrix. The remaining part the paper is devoted to the proof of this formula. In section 4 we describe the tools that are used for the derivation of the sum formula. In section 5 we evaluate a multiple successive action of the monodromy matrix entries onto Bethe vectors. Using this result we obtain the sum formulas for the scalar product in section 6. Appendices A and B contain the properties of the Izergin determinant and the highest coefficients. In appendix C we prove a summation identity for the highest coefficients.

2 General background

2.1 The model

The $GL(3)$ trigonometric quantum R -matrix has the following form

$$\begin{aligned} R(u, v) = & f(u, v) \sum_{1 \leq i \leq 3} \mathbf{E}_{ii} \otimes \mathbf{E}_{ii} + \sum_{1 \leq i < j \leq 3} (\mathbf{E}_{ii} \otimes \mathbf{E}_{jj} + \mathbf{E}_{jj} \otimes \mathbf{E}_{ii}) \\ & + \sum_{1 \leq i < j \leq 3} (u g(u, v) \mathbf{E}_{ij} \otimes \mathbf{E}_{ji} + v g(u, v) \mathbf{E}_{ji} \otimes \mathbf{E}_{ij}). \end{aligned} \quad (2.1)$$

Here the rational functions $f(u, v)$ and $g(u, v)$ are

$$f(u, v) = \frac{qu - q^{-1}v}{u - v}, \quad g(u, v) = \frac{q - q^{-1}}{u - v}, \quad (2.2)$$

where q is a complex number (a deformation parameter), and $(\mathbf{E}_{ij})_{lk} = \delta_{il}\delta_{jk}$, $i, j, l, k = 1, 2, 3$ are 3×3 matrices with unit in the intersection of i th row and j th column and zero matrix elements elsewhere.

In this paper we will consider quantum integrable models with the 3×3 monodromy matrix $T(u)$ which satisfies standard commutation relation (RTT -relation)

$$R(u, v) \cdot (T(u) \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes T(v)) = (\mathbf{1} \otimes T(v)) \cdot (T(u) \otimes \mathbf{1}) \cdot R(u, v). \quad (2.3)$$

The entries $T_{ij}(u)$ of the monodromy matrix form the quadratic algebra with commutation relations given by (2.3) and act in a quantum space V . We denote this algebra as \mathcal{A}_q and further will consider certain morphisms of \mathcal{A}_q (see (4.8)) onto $\mathcal{A}_{q^{-1}}$.¹ We assume that the vector space V possesses a highest weight vector $|0\rangle \in V$ such that

$$T_{ij}(u)|0\rangle = 0, \quad i > j, \quad T_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle, \quad \lambda_i(u) \in \mathbb{C}[[u, u^{-1}]]. \quad (2.4)$$

We also assume that the operators $T_{ij}(u)$ act in a dual space V^* with a vector $\langle 0| \in V^*$ such that

$$\langle 0|T_{ij}(u) = 0, \quad i < j, \quad \langle 0|T_{ii}(u) = \lambda_i(u)\langle 0|, \quad (2.5)$$

and λ_i are the same as in (2.4).

Below we permanently deal with sets of variables and their partitions into subsets. We denote sets of variables by bar: \bar{u} , \bar{v} and so on,

$$\bar{u} = \{u_1, \dots, u_a\}, \quad \bar{v} = \{v_1, \dots, v_b\}. \quad (2.6)$$

If necessary, the cardinalities of the sets will be described in special comments after the formulas.

If a set of variables is multiplied by a number $\alpha\bar{u}$ (in particular, $\bar{u}q^{\pm 2}$), then it means that all the elements of the set are multiplied by this number

$$\alpha\bar{u} = \{\alpha u_1, \dots, \alpha u_a\}, \quad \bar{v}q^{\pm 2} = \{v_1q^{\pm 2}, \dots, v_bq^{\pm 2}\}. \quad (2.7)$$

¹If monodromy matrix $T(u)$ is a generating series with respect to the parameter u^{-1} , the algebra \mathcal{A}_q can be identified with the positive Borel subalgebra in the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_3)$.

Union of sets is denoted by braces, for example, $\{\bar{w}, \bar{u}\} = \bar{\eta}$. Partitions of sets into disjoint subsets are denoted by the symbol \Rightarrow , and the subsets are numerated by roman numbers. For example, notation $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ means that the set \bar{u} is divided into two subsets \bar{u}_I and \bar{u}_{II} , such that $\bar{u}_I \cap \bar{u}_{II} = \emptyset$ and $\{\bar{u}_I, \bar{u}_{II}\} = \bar{u}$. Similarly, notation $\bar{\eta} \Rightarrow \{\bar{\eta}_i, \bar{\eta}_{ii}, \bar{\eta}_{iii}\}$ means that the set $\bar{\eta}$ is divided into three subsets with pair-wise empty intersections and $\{\bar{\eta}_i, \bar{\eta}_{ii}, \bar{\eta}_{iii}\} = \bar{\eta}$. Sometimes, when the number of subsets is big, we will use the standard arabic numbers for their numeration. In such cases we will give special comments.

Just like in the paper [5] we use shorthand notations for products with respect to sets of variables:

$$T_{ij}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{ij}(w_k); \quad \lambda_2(\bar{u}) = \prod_{u_j \in \bar{u}} \lambda_2(u_j); \quad r_k(\bar{v}) = \prod_{v_j \in \bar{v}} r_k(v_j), \quad (2.8)$$

where $r_k(w) = \lambda_k(w)/\lambda_2(w)$, $k = 1, 3$. That is, if the operator T_{ij} or functions λ_i and r_k depend on a set of variables, this means that one should take the product of the operators or the scalar functions with respect to the corresponding set. The same convention will be used for the products of functions $f(u, v)$

$$f(u, \bar{w}) = \prod_{w_j \in \bar{w}} f(u, w_j); \quad f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k). \quad (2.9)$$

The central object of the theory of scalar products in the $GL(2)$ -based models is the Izergin determinant $K_k(\bar{x}|\bar{y})$ [30]. It also plays an important role in the case of the models with $GL(3)$ trigonometric R -matrix. It is defined for two sets \bar{x} and \bar{y} of same cardinality $\#\bar{x} = \#\bar{y} = k$:

$$K_k(\bar{x}|\bar{y}) = \frac{\prod_{1 \leq i, j \leq k} (qx_i - q^{-1}y_j)}{\prod_{1 \leq i < j \leq k} (x_i - x_j)(y_j - y_i)} \cdot \det \left[\frac{q - q^{-1}}{(x_i - y_j)(qx_i - q^{-1}y_j)} \right]. \quad (2.10)$$

Below we also use two modifications of the Izergin determinant

$$K_k^{(l)}(\bar{x}|\bar{y}) = \prod_{i=1}^k x_i \cdot K_k(\bar{x}|\bar{y}), \quad K_k^{(r)}(\bar{x}|\bar{y}) = \prod_{i=1}^k y_i \cdot K_k(\bar{x}|\bar{y}), \quad (2.11)$$

which we call left and right Izergin determinants respectively. Some properties of the Izergin determinant and its modifications are gathered in appendix A.

Finally, we recall the formulas for the left and right highest coefficients of the scalar products [5]. These rational functions depend on four sets of variables and are denoted as $Z_{a,b}^{(l)}(\bar{t}; \bar{x}|\bar{s}; \bar{y})$ and $Z_{a,b}^{(r)}(\bar{t}; \bar{x}|\bar{s}; \bar{y})$ respectively. The subscripts show that $\#\bar{t} = \#\bar{x} = a$ and $\#\bar{s} = \#\bar{y} = b$. Both highest coefficients have representations in terms of the Izergin determinants

$$Z_{a,b}^{(l,r)}(\bar{t}; \bar{x}|\bar{s}; \bar{y}) = (-q)^{\mp b} \sum K_b^{(r,l)}(\bar{s}|\bar{w}_I q^2) K_a^{(l,r)}(\bar{w}_{II}|\bar{t}) K_b^{(l,r)}(\bar{y}|\bar{w}_I) f(\bar{w}_I, \bar{w}_{II}). \quad (2.12)$$

Here $\bar{w} = \{\bar{s}, \bar{x}\}$. The sum is taken with respect to partitions of the set $\bar{w} \Rightarrow \{\bar{w}_I, \bar{w}_{II}\}$ with $\#\bar{w}_I = b$ and $\#\bar{w}_{II} = a$. Recall also that according to the convention on the shorthand notations (2.9) the function $f(\bar{w}_I, \bar{w}_{II})$ actually means the double product of functions $f(\bar{w}_j, \bar{w}_k)$ over $w_j \in \bar{w}_I$ and $w_k \in \bar{w}_{II}$. The superscript (l, r) on $Z_{a,b}$ means that the equation (2.12) is valid

for $Z_{a,b}^{(l)}$ and for $Z_{a,b}^{(r)}$ separately. Choosing the first or the second component of (l, r) and the corresponding (up or down resp.) exponent of $(-q)^{\mp b}$ in this equation, we obtain representations either for $Z_{a,b}^{(l)}(\bar{t}; \bar{x}|\bar{s}; \bar{y})$ or for $Z_{a,b}^{(r)}(\bar{t}; \bar{x}|\bar{s}; \bar{y})$. Some properties of the highest coefficients are given in appendix B.

3 Scalar product of Bethe vectors

We recall that the entries of the monodromy matrix act on the space V , such that $|0\rangle \in V$. Other vectors of the space V can be obtained via the successive action of the creation operators $T_{12}(u)$, $T_{23}(u)$, $T_{13}(u)$ on the vector $|0\rangle$. Among all vectors of this space the Bethe vectors play the most important role. The procedure to construct the Bethe vectors was formulated in [31] in the framework of the nested algebraic Bethe ansatz (see also [32, 33, 34, 35, 36, 37, 38]). These vectors depend on complex variables, which are called the Bethe parameters. If the Bethe parameters satisfy the system of Bethe equations (see [31]), then the corresponding Bethe vector becomes an eigenstate of the transfer matrix $t(u) = \text{tr } T(u)$.

We denote the Bethe vectors by $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$. In quantum integrable models with a $GL(3)$ trigonometric R -matrix they depend on two sets of Bethe parameters \bar{u} and \bar{v} , such that $\#\bar{u} = a$, $\#\bar{v} = b$ and $a, b = 0, 1, \dots$.

Dual Bethe vectors $\mathbb{C}^{a,b}(\bar{u}; \bar{v})$ belong to the dual space V^* . They also depend on two sets of Bethe parameters \bar{u} and \bar{v} , with $\#\bar{u} = a$, $\#\bar{v} = b$ and $a, b = 0, 1, \dots$.

The scalar products are defined as

$$S_{a,b}(\bar{u}^C; \bar{v}^C | \bar{u}^B; \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (3.1)$$

where all the Bethe parameters are generic complex numbers. We have added the superscripts C and B to the sets \bar{u} , \bar{v} in order to stress that the vectors $\mathbb{C}^{a,b}$ and $\mathbb{B}^{a,b}$ may depend on different sets of parameters. The main result of this paper is a sum formula for the scalar product of Bethe vectors.

Proposition 3.1. *The scalar product of two Bethe vectors (3.1) is given by*

$$S_{a,b}(\bar{u}^C; \bar{v}^C | \bar{u}^B; \bar{v}^B) = \sum \frac{r_1(\bar{u}_\Pi^C) r_1(\bar{u}_\Pi^B) r_3(\bar{v}_\Pi^C) r_3(\bar{v}_\Pi^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} W_{\text{part}} \left(\begin{matrix} \bar{u}_\Pi^C, \bar{u}_\Pi^B, & \bar{u}_\text{I}^C, \bar{u}_\text{I}^B \\ \bar{v}_\text{I}^C, \bar{v}_\text{I}^B, & \bar{v}_\Pi^C, \bar{v}_\Pi^B \end{matrix} \right). \quad (3.2)$$

Here the sum runs over all the partitions $\bar{u}^C \Rightarrow \{\bar{u}_\text{I}^C, \bar{u}_\Pi^C\}$, $\bar{u}^B \Rightarrow \{\bar{u}_\text{I}^B, \bar{u}_\Pi^B\}$, $\bar{v}^C \Rightarrow \{\bar{v}_\text{I}^C, \bar{v}_\Pi^C\}$ and $\bar{v}^B \Rightarrow \{\bar{v}_\text{I}^B, \bar{v}_\Pi^B\}$ with $\#\bar{u}_\text{I}^C = \#\bar{u}_\text{I}^B$ and $\#\bar{v}_\text{I}^C = \#\bar{v}_\text{I}^B$. For a fixed partition with $\#\bar{u}_\text{I}^C = \#\bar{u}_\text{I}^B = k$ and $\#\bar{v}_\text{I}^C = \#\bar{v}_\text{I}^B = n$, (where $k = 0, \dots, a$ and $n = 0, \dots, b$), the rational coefficient W_{part} has the form

$$\begin{aligned} W_{\text{part}} \left(\begin{matrix} \bar{u}_\Pi^C, \bar{u}_\Pi^B, & \bar{u}_\text{I}^C, \bar{u}_\text{I}^B \\ \bar{v}_\text{I}^C, \bar{v}_\text{I}^B, & \bar{v}_\Pi^C, \bar{v}_\Pi^B \end{matrix} \right) &= f(\bar{u}_\Pi^B, \bar{u}_\text{I}^B) f(\bar{u}_\text{I}^C, \bar{u}_\Pi^C) f(\bar{v}_\text{I}^B, \bar{v}_\Pi^B) f(\bar{v}_\Pi^C, \bar{v}_\text{I}^C) f(\bar{v}_\text{I}^C, \bar{u}_\text{I}^C) f(\bar{v}_\Pi^B, \bar{u}_\Pi^B) \\ &\times Z_{a-k,n}^{(l)}(\bar{u}_\Pi^C; \bar{v}_\Pi^B | \bar{v}_\text{I}^C; \bar{v}_\text{I}^B) Z_{k,b-n}^{(r)}(\bar{u}_\text{I}^B; \bar{u}_\text{I}^C | \bar{v}_\Pi^B; \bar{v}_\Pi^C). \end{aligned} \quad (3.3)$$

where the highest coefficients $Z^{(l,r)}$ are given by (2.12).

In the work [5] we have found the rational functions W_{part} corresponding to the extreme partitions

$$\begin{aligned} W_{\text{part}} \begin{pmatrix} \bar{u}^C, \bar{u}^B, & \emptyset, \emptyset \\ \bar{v}^C, \bar{v}^B, & \emptyset, \emptyset \end{pmatrix} &= Z_{a,b}^{(l)}(\bar{u}^C; \bar{u}^B | \bar{v}^C; \bar{v}^B), \\ W_{\text{part}} \begin{pmatrix} \emptyset, \emptyset, & \bar{u}^C, \bar{u}^B \\ \emptyset, \emptyset, & \bar{v}^C, \bar{v}^B \end{pmatrix} &= Z_{a,b}^{(r)}(\bar{u}^B; \bar{u}^C | \bar{v}^B; \bar{v}^C). \end{aligned} \quad (3.4)$$

Proposition 3.1 determines the functions W_{part} for arbitrary partitions of the Bethe parameters. In the following, we prove proposition 3.1.

4 Necessary tools

In this section we describe the tools that we use for the calculation of the scalar products. As we have already explained, our method of calculation is based on the explicit formulas for the dual Bethe vectors and for the multiple actions of the monodromy matrix entries onto Bethe vectors. In this way one can obtain a representation for the scalar product as a sum over partitions of Bethe parameters. In order to simplify this representation we use a special isomorphism between \mathcal{A}_q and $\mathcal{A}_{q^{-1}}$ algebras.

4.1 Explicit representations for dual Bethe vectors

We use explicit representations for dual Bethe vectors $\mathbb{C}^{a,b}(\bar{u}; \bar{v})$, which were found in [38]²

$$\mathbb{C}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{(-q)^k \mathbf{K}_k^{(l)}(\bar{u}_I | q^2 \bar{v}_I)}{\lambda_2(\bar{v}_I) \lambda_2(\bar{u}) f(\bar{v}_I, \bar{u})} f(\bar{v}_I, \bar{v}_I) f(\bar{u}_I, \bar{u}_I) \langle 0 | T_{32}(\bar{v}_I) T_{31}(\bar{u}_I) T_{21}(\bar{u}_I), \quad (4.1)$$

and

$$\mathbb{C}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{(-q)^{-k} \mathbf{K}_k^{(r)}(\bar{u}_I | q^2 \bar{v}_I)}{\lambda_2(\bar{v}) \lambda_2(\bar{u}_I) f(\bar{v}, \bar{u}_I)} f(\bar{v}_I, \bar{v}_I) f(\bar{u}_I, \bar{u}_I) \langle 0 | T_{21}(\bar{u}_I) T_{31}(\bar{v}_I) T_{32}(\bar{v}_I). \quad (4.2)$$

Here the sum goes over all partitions of the sets $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_I\}$ and $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_I\}$ such that $\#\bar{u}_I = \#\bar{v}_I = k$, $k = 0, \dots, \min(a, b)$. Both of these representations are needed for our purpose. In section 4.3 we will show that (4.1) and (4.2) also are related by an isomorphism φ between \mathcal{A}_q and $\mathcal{A}_{q^{-1}}$ algebras.

4.2 Multiple actions of T_{ij} on Bethe vectors

In order to compute the scalar product we need formulas of the multiple action of the operators T_{ij} with $i > j$ onto the Bethe vectors. These multiple actions were derived in the work [39] (see also [5]). We give here the list of necessary formulas, including some important particular cases. Below everywhere in this section $\#\bar{w} = n$, $\{\bar{w}, \bar{u}\} = \bar{\eta}$, $\{\bar{w}, \bar{v}\} = \bar{\xi}$.

²In order to reduce (4.1), (4.2) to the representations of [38] one should apply (A.4) to the Izergin determinants in these formulas.

• **Multiple action of T_{21} .**

$$T_{21}(\bar{w})\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = (-q)^n \lambda_2(\bar{w}) \sum r_1(\bar{\eta}_I) f(\bar{\eta}_I, \bar{\eta}_I) f(\bar{\eta}_I, \bar{\eta}_I) f(\bar{\eta}_I, \bar{\eta}_I) \frac{f(\bar{\xi}_I, \bar{\xi}_I)}{f(\bar{\xi}_I, \bar{\eta}_I)} \\ \times K_n^{(r)}(q^{-2}\bar{w}|\bar{\eta}_I) K_n^{(l)}(\bar{\eta}_I|q^2\bar{\xi}_I) K_n^{(l)}(\bar{\xi}_I|q^2\bar{w}) \mathbb{B}^{a-n,b}(\bar{\eta}_I; \bar{\xi}_I). \quad (4.3)$$

The sum is taken over partitions of: $\bar{\eta} \Rightarrow \{\bar{\eta}_I, \bar{\eta}_I, \bar{\eta}_I\}$ with $\#\bar{\eta}_I = \#\bar{\eta}_I = n$; and $\bar{\xi} \Rightarrow \{\bar{\xi}_I, \bar{\xi}_I\}$ with $\#\bar{\xi}_I = n$.

• **Multiple action of T_{31} .**

$$T_{31}(\bar{w})\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{w}) \sum r_1(\bar{\eta}_I) r_3(\bar{\xi}_I) K_n^{(r)}(q^{-2}\bar{\eta}_I|\bar{\xi}_I) K_n^{(l)}(\bar{\eta}_I|q^2\bar{\xi}_I) K_n^{(r)}(q^{-2}\bar{w}|\bar{\eta}_I) K_n^{(l)}(\bar{\xi}_I|q^2\bar{w}) \\ \times \frac{f(\bar{\eta}_I, \bar{\eta}_I) f(\bar{\eta}_I, \bar{\eta}_I) f(\bar{\eta}_I, \bar{\eta}_I) f(\bar{\xi}_I, \bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I)}{f(\bar{\xi}_I, \bar{\eta}_I) f(\bar{\xi}_I, \bar{\eta}_I) f(\bar{\xi}_I, \bar{\eta}_I)} \mathbb{B}^{a-n,b-n}(\bar{\eta}_I; \bar{\xi}_I). \quad (4.4)$$

The sum is taken over partitions of: $\bar{\xi} \Rightarrow \{\bar{\xi}_I, \bar{\xi}_I, \bar{\xi}_I\}$ with $\#\bar{\xi}_I = \#\bar{\xi}_I = n$; $\bar{\eta} \Rightarrow \{\bar{\eta}_I, \bar{\eta}_I, \bar{\eta}_I\}$ with $\#\bar{\eta}_I = \#\bar{\eta}_I = n$.

Remark that in the particular case $a = n$, we have $\bar{\eta}_I = \emptyset$, and then

$$T_{31}(\bar{w})\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{w}) \sum r_1(\bar{\eta}_I) r_3(\bar{\xi}_I) K_n^{(r)}(q^{-2}\bar{\eta}_I|\bar{\xi}_I) K_n^{(l)}(\bar{\eta}_I|q^2\bar{\xi}_I) K_n^{(r)}(q^{-2}\bar{w}|\bar{\eta}_I) K_n^{(l)}(\bar{\xi}_I|q^2\bar{w}) \\ \times \frac{f(\bar{\eta}_I, \bar{\eta}_I) f(\bar{\xi}_I, \bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I)}{f(\bar{\xi}_I, \bar{\eta}_I) f(\bar{\xi}_I, \bar{\eta}_I)} \mathbb{B}^{0,b-n}(\emptyset; \bar{\xi}_I). \quad (4.5)$$

• **Multiple action of T_{32} .**

$$T_{32}(\bar{w})\mathbb{B}^{a,b}(\bar{u}; \bar{v}) = (-q)^{-n} \lambda_2(\bar{w}) \sum r_3(\bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I) \frac{f(\bar{\eta}_I, \bar{\eta}_I)}{f(\bar{\xi}_I, \bar{\eta}_I)} \\ + \times K_n^{(r)}(q^{-2}\bar{w}|\bar{\eta}_I) K_n^{(r)}(q^{-2}\bar{\eta}_I|\bar{\xi}_I) K_n^{(l)}(\bar{\xi}_I|q^2\bar{w}) \mathbb{B}^{a,b-n}(\bar{\eta}_I; \bar{\xi}_I). \quad (4.6)$$

The sum is taken over partitions of: $\bar{\xi} \Rightarrow \{\bar{\xi}_I, \bar{\xi}_I, \bar{\xi}_I\}$ with $\#\bar{\xi}_I = \#\bar{\xi}_I = n$; and $\bar{\eta} \Rightarrow \{\bar{\eta}_I, \bar{\eta}_I\}$ with $\#\bar{\eta}_I = n$.

Note that in the special case $a = 0$, we have $\bar{\eta}_I = \bar{w}$ and $\bar{\eta}_I = \emptyset$. If in addition $b = n$, then $\bar{\xi}_I = \emptyset$ and we obtain

$$T_{32}(\bar{w})\mathbb{B}^{0,b}(\emptyset; \bar{v}) = \lambda_2(\bar{w}) \sum r_3(\bar{\xi}_I) f(\bar{\xi}_I, \bar{\xi}_I) K_n^{(r)}(q^{-2}\bar{w}|\bar{\xi}_I) K_n^{(l)}(\bar{\xi}_I|q^2\bar{w}) |0\rangle. \quad (4.7)$$

Remark. In all formulas for the multiple actions, we described the cardinalities of subsets in special comments. Actually these comments are not necessary, since the cardinalities of the subsets are shown explicitly directly in the formulas. Indeed, the subscript of the Izergin determinant indicates the number of elements in both sets of variables on which it depends. Hence, looking, for example, at equation (4.3) we conclude that $\#\bar{\eta}_I = \#\bar{\eta}_I = \#\bar{\xi}_I = \#\bar{w} = n$. In addition, the superscripts of Bethe vector $\mathbb{B}^{a-n,b}(\bar{\eta}_I; \bar{\xi}_I)$ show that $\#\bar{\eta}_I = a - n$ and $\#\bar{\xi}_I = b$. For the reader convenience, below we will continue to give separate comments about the cardinalities of subsets. However in equations containing a big number of subsets we will skip such descriptions.

4.3 Isomorphism between \mathcal{A}_q and $\mathcal{A}_{q^{-1}}$ algebras

In the work [38] we have described an isomorphism between positive Borel subalgebras in the algebras $U_q(\mathfrak{gl}_N)$ and $U_{q^{-1}}(\mathfrak{gl}_N)$. We denote this map by φ . In the case of the algebras \mathcal{A}_q and $\mathcal{A}_{q^{-1}}$ the map φ has the form

$$\varphi(T_{i,j}(u)) = \tilde{T}_{4-j,4-i}(u), \quad (4.8)$$

where $T_{i,j}(u) \in \mathcal{A}_q$ and $\tilde{T}_{4-j,4-i}(u) \in \mathcal{A}_{q^{-1}}$, respectively. The action of φ on Bethe vectors is given by (see [38]):

$$\varphi(\mathbb{B}^{a,b}(\bar{u}; \bar{v})) = \mathbb{B}_{q^{-1}}^{b,a}(\bar{v}; \bar{u}), \quad \varphi(\mathbb{C}^{a,b}(\bar{u}; \bar{v})) = \mathbb{C}_{q^{-1}}^{b,a}(\bar{v}; \bar{u}). \quad (4.9)$$

Here we have equipped the Bethe vector and its dual by the additional subscript q^{-1} in order to stress that these vectors are constructed for the algebra $\mathcal{A}_{q^{-1}}$. Generically, the action of φ can be described as follows

$$\varphi(\mathcal{F}(T_{i,j}(u); \lambda_i(u); q)) = \mathcal{F}(\tilde{T}_{4-j,4-i}(u); \tilde{\lambda}_{4-i}(u); q^{-1}), \quad \text{where } \tilde{\lambda}_i(u) = \langle 0 | \tilde{T}_{ii}(u) | 0 \rangle. \quad (4.10)$$

Here \mathcal{F} is some polynomial in the operators $T_{ij}(u)$, whose coefficients may also depend on the vacuum eigenvalues $\lambda_i(u)$ and the parameter q .

The map (4.8) is a very powerful tool for the study of the scalar products. We will use it in section 6.2. Here we use this isomorphism in order to show the equivalence of the representations (4.1) and (4.2) for the dual Bethe vectors.

We start with the representation (4.1), written in $\mathcal{A}_{q^{-1}}$:

$$\mathbb{C}_{q^{-1}}^{b,a}(\bar{v}; \bar{u}) = \sum \frac{(-q)^{-k} \mathbb{K}_{k,q^{-1}}^{(l)}(\bar{v}_I | q^{-2} \bar{u}_I)}{\tilde{\lambda}_2(\bar{u}_\Pi) \tilde{\lambda}_2(\bar{v}) f_{q^{-1}}(\bar{u}_\Pi, \bar{u}_I)} f_{q^{-1}}(\bar{u}_\Pi, \bar{u}_I) f_{q^{-1}}(\bar{v}_\Pi, \bar{v}_I) \langle 0 | \tilde{T}_{32}(\bar{u}_\Pi) \tilde{T}_{31}(\bar{v}_I) \tilde{T}_{21}(\bar{v}_\Pi), \quad (4.11)$$

Here, similarly to (4.9) we have written an additional subscripts q^{-1} to the function f and the Izergin determinant. This means that $f_{q^{-1}}$ is given by (2.2) with q replaced by q^{-1} . Similarly $\mathbb{K}_{k,q^{-1}}^{(l,r)}$ are given by (2.11) and (2.10), where q is replaced by q^{-1} .

It is easy to check that

$$f_{q^{-1}}(x, y) = f(y, x), \quad \mathbb{K}_{k,q^{-1}}^{(l,r)}(\bar{x} | \bar{y}) = \mathbb{K}_k^{(r,l)}(\bar{y} | \bar{x}). \quad (4.12)$$

Then one can recast (4.11) as follows

$$\varphi(\mathbb{C}^{a,b}(\bar{u}; \bar{v})) = \sum \frac{(-q)^{-k} \mathbb{K}_k^{(r)}(q^{-2} \bar{v}_I | \bar{u}_I)}{\tilde{\lambda}_2(\bar{u}_\Pi) \tilde{\lambda}_2(\bar{v}) f(\bar{v}, \bar{u}_\Pi)} f(\bar{u}_I, \bar{u}_\Pi) f(\bar{v}_I, \bar{v}_\Pi) \langle 0 | \tilde{T}_{32}(\bar{u}_\Pi) \tilde{T}_{31}(\bar{v}_I) \tilde{T}_{21}(\bar{v}_\Pi), \quad (4.13)$$

where we have used (4.9). Now we apply the mapping φ directly to the representation (4.13) using (4.10). Recall that φ acts only on the operators \tilde{T}_{ij} and the vacuum eigenvalue $\tilde{\lambda}_2$. Taking into account that $\varphi^2 = 1$ we obtain

$$\mathbb{C}^{a,b}(\bar{u}; \bar{v}) = \sum \frac{(-q)^{-k} \mathbb{K}_k^{(r)}(\bar{v}_I | \bar{u}_I q^2)}{\lambda_2(\bar{u}_\Pi) \lambda_2(\bar{v}) f(\bar{v}, \bar{u}_\Pi)} f(\bar{u}_I, \bar{u}_\Pi) f(\bar{v}_I, \bar{v}_\Pi) \langle 0 | T_{21}(\bar{u}_\Pi) T_{31}(\bar{v}_I) T_{32}(\bar{v}_\Pi), \quad (4.14)$$

where we used $\mathbb{K}_k^{(r)}(q^{-2} \bar{x} | \bar{y}) = \mathbb{K}_k^{(r)}(\bar{x} | \bar{y} q^2)$ (see (A.2)). Thus, we have reproduced the representation (4.2).

5 Successive action

The first step in the derivation of the scalar product is to find the result of the multiple successive action of the operators $T_{32}(\bar{z})T_{31}(\bar{y})T_{21}(\bar{x})$ onto generic Bethe vector $\mathbb{B}^{a,b}(\bar{u}, \bar{v})$, where \bar{z} , \bar{y} , \bar{x} , \bar{u} , and \bar{v} are sets of arbitrary complex numbers. It follows from the explicit representation of the dual Bethe vector (4.1) that for our goal it is enough to consider the case $\#\bar{y} + \#\bar{x} = \#\bar{u}$ and $\#\bar{y} + \#\bar{z} = \#\bar{v}$. Therefore we set³ $\#\bar{y} = k$, $\#\bar{x} = a - k$, and $\#\bar{z} = b - k$, where $k = 0, 1, \dots, \min(a, b)$.

For the derivation of the multiple successive action we use the formulas (4.3), (4.5), (4.7). It is not difficult to guess that the final result will contain a sum over partitions of the original sets of variables into a big number of subsets. Therefore, in order to avoid cumbersome roman numbers we use in this section standard arabic numbers for notations of subsets.

5.1 Successive action of $T_{31}(\bar{y})T_{21}(\bar{x})$

Let

$$S_1 = T_{31}(\bar{y})T_{21}(\bar{x})\mathbb{B}^{a,b}(\bar{u}; \bar{v}). \quad (5.1)$$

The action of $T_{21}(\bar{x})$ is given by (4.3), where one should set $\bar{w} = \bar{x}$ and $n = a - k$. We obtain

$$S_1 = (-q)^{a-k} \lambda_2(\bar{x}) T_{31}(\bar{y}) \sum r_1(\bar{\eta}_1) f(\bar{\eta}_2, \bar{\eta}_1) f(\bar{\eta}_2, \bar{\eta}_3) f(\bar{\eta}_3, \bar{\eta}_1) \frac{f(\bar{\xi}_2, \bar{\xi}_1)}{f(\bar{\xi}_2, \bar{\eta}_1)} \\ \times K_{a-k}^{(r)}(q^{-2}\bar{x}|\bar{\eta}_2) K_{a-k}^{(l)}(\bar{\eta}_1|q^2\bar{\xi}_1) K_{a-k}^{(l)}(\bar{\xi}_1|q^2\bar{x}) \mathbb{B}^{k,b}(\bar{\eta}_3; \bar{\xi}_2), \quad (5.2)$$

where $\eta = \{\bar{u}, \bar{x}\}$ and $\xi = \{\bar{v}, \bar{x}\}$. Recall that the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_1, \bar{\xi}_2\}$. The cardinalities of the subsets are equal to the subscripts of the corresponding Izergin determinants or to the superscripts of the Bethe vector (see Remark on page 7).

The action of $T_{31}(\bar{y})$ creates new partitions $\{\bar{y}, \bar{\eta}_3\} \Rightarrow \{\bar{\eta}_4, \bar{\eta}_5\}$ and $\{\bar{y}, \bar{\xi}_2\} \Rightarrow \{\bar{\xi}_3, \bar{\xi}_4, \bar{\xi}_5\}$. It means, in particular, that the products over subset $\bar{\eta}_3$ in (5.2) should be replaced by

$$f(\bar{\eta}_2, \bar{\eta}_3) f(\bar{\eta}_3, \bar{\eta}_1) = \frac{f(\bar{\eta}_2, \bar{\eta}_4) f(\bar{\eta}_2, \bar{\eta}_5) f(\bar{\eta}_5, \bar{\eta}_1) f(\bar{\eta}_4, \bar{\eta}_1)}{f(\bar{\eta}_2, \bar{y}) f(\bar{y}, \bar{\eta}_1)}. \quad (5.3)$$

Similarly

$$\frac{f(\bar{\xi}_2, \bar{\xi}_1)}{f(\bar{\xi}_2, \bar{\eta}_1)} = \frac{f(\bar{\xi}_3, \bar{\xi}_1) f(\bar{\xi}_4, \bar{\xi}_1) f(\bar{\xi}_5, \bar{\xi}_1) f(\bar{y}, \bar{\eta}_1)}{f(\bar{\xi}_3, \bar{\eta}_1) f(\bar{\xi}_4, \bar{\eta}_1) f(\bar{\xi}_5, \bar{\eta}_1) f(\bar{y}, \bar{\xi}_1)}. \quad (5.4)$$

³Recall that the cardinalities of the sets \bar{u} and \bar{v} coincide with the superscripts of the Bethe vector a and b respectively.

Then using (4.5) we obtain

$$\begin{aligned}
S_1 = & (-q)^{a-k} \lambda_2(\bar{x}) \lambda_2(\bar{y}) \sum r_1(\bar{\eta}_1) r_1(\bar{\eta}_5) r_3(\bar{\xi}_3) \mathsf{K}_{a-k}^{(r)}(q^{-2}\bar{x}|\bar{\eta}_2) \mathsf{K}_{a-k}^{(l)}(\bar{\eta}_1|q^2\bar{\xi}_1) \\
& \times \mathsf{K}_{a-k}^{(l)}(\bar{\xi}_1|q^2\bar{x}) \mathsf{K}_k^{(r)}(q^{-2}\bar{\eta}_4|\bar{\xi}_3) \mathsf{K}_k^{(l)}(\bar{\eta}_5|q^2\bar{\xi}_4) \mathsf{K}_k^{(r)}(q^{-2}\bar{y}|\bar{\eta}_4) \mathsf{K}_k^{(l)}(\bar{\xi}_4|q^2\bar{y}) \\
& \times f(\bar{\eta}_2, \bar{\eta}_4) f(\bar{\eta}_2, \bar{\eta}_1) f(\bar{\eta}_2, \bar{\eta}_5) f(\bar{\eta}_4, \bar{\eta}_5) f(\bar{\eta}_4, \bar{\eta}_1) f(\bar{\eta}_5, \bar{\eta}_1) \\
& \times \frac{f(\bar{\xi}_3, \bar{\xi}_4) f(\bar{\xi}_3, \bar{\xi}_1) f(\bar{\xi}_3, \bar{\xi}_5) f(\bar{\xi}_5, \bar{\xi}_4) f(\bar{\xi}_5, \bar{\xi}_1) f(\bar{\xi}_4, \bar{\xi}_1)}{f(\bar{\xi}_3, \bar{\eta}_1) f(\bar{\xi}_3, \bar{\eta}_5) f(\bar{\xi}_5, \bar{\eta}_1) f(\bar{\xi}_5, \bar{\eta}_5) f(\bar{\xi}_4, \bar{\eta}_1) f(\bar{y}, \bar{\xi}_1) f(\bar{\eta}_2, \bar{y})} \mathbb{B}^{0, b-k}(\emptyset; \bar{\xi}_5). \quad (5.5)
\end{aligned}$$

Here $\eta = \{\bar{u}, \bar{x}, \bar{y}\}$ and $\xi = \{\bar{v}, \bar{x}, \bar{y}\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4, \bar{\eta}_5\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4, \bar{\xi}_5\}$.

The expression (5.5) can be slightly simplified. First of all we set $\{\bar{\eta}_1, \bar{\eta}_5\} = \bar{\eta}_6$ and $\{\bar{\xi}_1, \bar{\xi}_4\} = \bar{\xi}_6$. Then the equation (5.5) takes the form

$$\begin{aligned}
S_1 = & (-q)^{a-k} \lambda_2(\bar{x}) \lambda_2(\bar{y}) \sum r_1(\bar{\eta}_6) r_3(\bar{\xi}_3) \left\{ \mathsf{K}_{a-k}^{(l)}(\bar{\eta}_1|q^2\bar{\xi}_1) \mathsf{K}_k^{(l)}(\bar{\eta}_5|q^2\bar{\xi}_4) \frac{f(\bar{\eta}_5, \bar{\eta}_1)}{f(\bar{\xi}_4, \bar{\eta}_1)} \right\} \\
& \times \mathsf{K}_{a-k}^{(r)}(q^{-2}\bar{x}|\bar{\eta}_2) \mathsf{K}_{a-k}^{(l)}(\bar{\xi}_1|q^2\bar{x}) \mathsf{K}_k^{(r)}(q^{-2}\bar{\eta}_4|\bar{\xi}_3) \mathsf{K}_k^{(r)}(q^{-2}\bar{y}|\bar{\eta}_4) \mathsf{K}_k^{(l)}(\bar{\xi}_4|q^2\bar{y}) \\
& \times \frac{f(\bar{\eta}_2, \bar{\eta}_4) f(\bar{\eta}_2, \bar{\eta}_6) f(\bar{\eta}_4, \bar{\eta}_6) f(\bar{\xi}_3, \bar{\xi}_6) f(\bar{\xi}_3, \bar{\xi}_5) f(\bar{\xi}_5, \bar{\xi}_6) f(\bar{\xi}_4, \bar{\xi}_1)}{f(\bar{\xi}_3, \bar{\eta}_6) f(\bar{\xi}_5, \bar{\eta}_6) f(\bar{y}, \bar{\xi}_1) f(\bar{\eta}_2, \bar{y})} \mathbb{B}^{0, b-k}(\emptyset; \bar{\xi}_5). \quad (5.6)
\end{aligned}$$

Now we can take the sum over the partitions $\bar{\eta}_6 \Rightarrow \{\bar{\eta}_1, \bar{\eta}_5\}$ (see the terms in the braces in (5.6)):

$$\begin{aligned}
& \sum \mathsf{K}_{a-k}^{(l)}(\bar{\eta}_1|q^2\bar{\xi}_1) \mathsf{K}_k^{(l)}(\bar{\eta}_5|q^2\bar{\xi}_4) \frac{f(\bar{\eta}_5, \bar{\eta}_1)}{f(\bar{\xi}_4, \bar{\eta}_1)} \\
& = (-q)^{-k} \sum \mathsf{K}_{a-k}^{(l)}(\bar{\eta}_1|q^2\bar{\xi}_1) \mathsf{K}_k^{(r)}(\bar{\xi}_4|\bar{\eta}_5) \frac{f(\bar{\eta}_5, \bar{\eta}_1)}{f(\bar{\xi}_4, \bar{\eta}_6)} = \mathsf{K}_a^{(l)}(\bar{\eta}_6|q^2\bar{\xi}_6). \quad (5.7)
\end{aligned}$$

Here we first transformed $\mathsf{K}_k^{(l)}$ into $\mathsf{K}_k^{(r)}$ via (A.4) and then applied (A.7). Thus, the equation (5.6) turns into

$$\begin{aligned}
S_1 = & (-q)^{a-k} \lambda_2(\bar{x}) \lambda_2(\bar{y}) \sum r_1(\bar{\eta}_6) r_3(\bar{\xi}_3) \mathsf{K}_{a-k}^{(r)}(q^{-2}\bar{x}|\bar{\eta}_2) \mathsf{K}_a^{(l)}(\bar{\eta}_6|q^2\bar{\xi}_6) \\
& \times \mathsf{K}_k^{(r)}(q^{-2}\bar{\eta}_4|\bar{\xi}_3) \mathsf{K}_k^{(r)}(q^{-2}\bar{y}|\bar{\eta}_4) \left\{ \mathsf{K}_{a-k}^{(l)}(\bar{\xi}_1|q^2\bar{x}) \mathsf{K}_k^{(l)}(\bar{\xi}_4|q^2\bar{y}) \frac{f(\bar{\xi}_4, \bar{\xi}_1)}{f(\bar{y}, \bar{\xi}_1)} \right\} \\
& \times \frac{f(\bar{\eta}_2, \bar{\eta}_4) f(\bar{\eta}_2, \bar{\eta}_6) f(\bar{\eta}_4, \bar{\eta}_6) f(\bar{\xi}_3, \bar{\xi}_6) f(\bar{\xi}_3, \bar{\xi}_5) f(\bar{\xi}_5, \bar{\xi}_6)}{f(\bar{\xi}_3, \bar{\eta}_6) f(\bar{\xi}_5, \bar{\eta}_6) f(\bar{\eta}_2, \bar{y})} \mathbb{B}^{0, b-k}(\emptyset; \bar{\xi}_5). \quad (5.8)
\end{aligned}$$

Now we can take the sum over partitions $\bar{\xi}_6 \Rightarrow \{\bar{\xi}_1, \bar{\xi}_4\}$ (see the terms in the braces in (5.8)):

$$\begin{aligned}
& \sum \mathsf{K}_{a-k}^{(l)}(\bar{\xi}_1|q^2\bar{x}) \mathsf{K}_k^{(l)}(\bar{\xi}_4|q^2\bar{y}) \frac{f(\bar{\xi}_4, \bar{\xi}_1)}{f(\bar{y}, \bar{\xi}_1)} \\
& = (-q)^{-k} \sum \mathsf{K}_{a-k}^{(l)}(\bar{\xi}_1|q^2\bar{x}) \mathsf{K}_k^{(r)}(\bar{y}|\bar{\xi}_4) \frac{f(\bar{\xi}_4, \bar{\xi}_1)}{f(\bar{y}, \bar{\xi}_6)} = \mathsf{K}_a^{(l)}(\bar{\xi}_6|\{q^2\bar{x}, q^2\bar{y}\}). \quad (5.9)
\end{aligned}$$

We again transformed $K_k^{(l)}$ into $K_k^{(r)}$ via (A.4) and applied (A.7). Thus, we finally arrive at

$$S_1 = (-q)^{a-k} \lambda_2(\bar{x}) \lambda_2(\bar{y}) \sum r_1(\bar{\eta}_6) r_3(\bar{\xi}_3) K_a^{(l)}(\bar{\xi}_6 | \{q^2 \bar{x}, q^2 \bar{y}\}) K_a^{(l)}(\bar{\eta}_6 | q^2 \bar{\xi}_6) K_{a-k}^{(r)}(q^{-2} \bar{x} | \bar{\eta}_2) \\ \times K_k^{(r)}(q^{-2} \bar{\eta}_4 | \bar{\xi}_3) K_k^{(r)}(q^{-2} \bar{y} | \bar{\eta}_4) \frac{f(\bar{\eta}_2, \bar{\eta}_4) f(\bar{\eta}_2, \bar{\eta}_6) f(\bar{\eta}_4, \bar{\eta}_6) f(\bar{\xi}_3, \bar{\xi}_5) f(\bar{\xi}_5, \bar{\xi}_6) f(\bar{\xi}_3, \bar{\xi}_6)}{f(\bar{\xi}_3, \bar{\eta}_6) f(\bar{\xi}_5, \bar{\eta}_6) f(\bar{\eta}_2, \bar{y})} \mathbb{B}^{0, b-k}(\emptyset; \bar{\xi}_5). \quad (5.10)$$

In this formula $\eta = \{\bar{u}, \bar{x}, \bar{y}\}$ and $\xi = \{\bar{v}, \bar{x}, \bar{y}\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_2, \bar{\eta}_4, \bar{\eta}_6\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_3, \bar{\xi}_5, \bar{\xi}_6\}$. The partitions are independent except their cardinalities that are fixed by the subscripts of the Izergin determinants and the superscript of Bethe vector.

5.2 Successive action of $T_{32}(\bar{z})T_{31}(\bar{y})T_{21}(\bar{x})$

For the computation of the successive action $T_{32}(\bar{z})T_{31}(\bar{y})T_{21}(\bar{x})$ on the Bethe vector we should act with the product $T_{32}(\bar{z})$ on S_1 . Recall that $\# \bar{z} = b - k$, therefore we can use (4.7). Then the action of $T_{32}(\bar{z})$ gives us an additional sum over partitions $\{\bar{z}, \bar{\xi}_5\} \Rightarrow \{\bar{\xi}_7, \bar{\xi}_8\}$. We have

$$T_{32}(\bar{z})S_1 = (-q)^{a-k} \lambda_2(\bar{x}) \lambda_2(\bar{y}) \lambda_2(\bar{z}) \sum r_1(\bar{\eta}_6) r_3(\bar{\xi}_3) r_3(\bar{\xi}_7) K_a^{(l)}(\bar{\xi}_6 | \{q^2 \bar{x}, q^2 \bar{y}\}) K_a^{(l)}(\bar{\eta}_6 | q^2 \bar{\xi}_6) \\ \times K_{a-k}^{(r)}(q^{-2} \bar{x} | \bar{\eta}_2) K_k^{(r)}(q^{-2} \bar{\eta}_4 | \bar{\xi}_3) K_k^{(r)}(q^{-2} \bar{y} | \bar{\eta}_4) K_{b-k}^{(r)}(q^{-2} \bar{z} | \bar{\xi}_7) K_{b-k}^{(l)}(\bar{\xi}_8 | q^2 \bar{z}) \\ \times \frac{f(\bar{\eta}_2, \bar{\eta}_4) f(\bar{\eta}_2, \bar{\eta}_6) f(\bar{\eta}_4, \bar{\eta}_6) f(\bar{\xi}_3, \bar{\xi}_7) f(\bar{\xi}_3, \bar{\xi}_8) f(\bar{\xi}_7, \bar{\xi}_8) f(\bar{\xi}_8, \bar{\xi}_6) f(\bar{\xi}_3, \bar{\xi}_6) f(\bar{\xi}_7, \bar{\xi}_6) f(\bar{z}, \bar{\eta}_6)}{f(\bar{\xi}_3, \bar{\eta}_6) f(\bar{\xi}_7, \bar{\eta}_6) f(\bar{\xi}_3, \bar{z}) f(\bar{z}, \bar{\xi}_6) f(\bar{\xi}_8, \bar{\eta}_6) f(\bar{\eta}_2, \bar{y})} |0\rangle. \quad (5.11)$$

In this formula we still have $\eta = \{\bar{u}, \bar{x}, \bar{y}\}$, but $\xi = \{\bar{v}, \bar{x}, \bar{y}, \bar{z}\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_2, \bar{\eta}_4, \bar{\eta}_6\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_3, \bar{\xi}_6, \bar{\xi}_7, \bar{\xi}_8\}$.

Again partial summation over partitions is possible. Setting $\{\bar{\xi}_3, \bar{\xi}_7\} = \bar{\xi}_9$ we obtain

$$T_{32}(\bar{z})S_1 = (-q)^{a-k} \lambda_2(\bar{x}) \lambda_2(\bar{y}) \lambda_2(\bar{z}) \sum r_1(\bar{\eta}_6) r_3(\bar{\xi}_9) K_a^{(l)}(\bar{\xi}_6 | \{q^2 \bar{x}, q^2 \bar{y}\}) K_a^{(l)}(\bar{\eta}_6 | q^2 \bar{\xi}_6) \\ \times K_{a-k}^{(r)}(q^{-2} \bar{x} | \bar{\eta}_2) K_k^{(r)}(q^{-2} \bar{y} | \bar{\eta}_4) K_{b-k}^{(l)}(\bar{\xi}_8 | q^2 \bar{z}) \left\{ K_k^{(r)}(q^{-2} \bar{\eta}_4 | \bar{\xi}_3) K_{b-k}^{(r)}(q^{-2} \bar{z} | \bar{\xi}_7) \frac{f(\bar{\xi}_3, \bar{\xi}_7)}{f(\bar{\xi}_3, \bar{z})} \right\} \\ \times \frac{f(\bar{\eta}_2, \bar{\eta}_4) f(\bar{\eta}_2, \bar{\eta}_6) f(\bar{\eta}_4, \bar{\eta}_6) f(\bar{\xi}_9, \bar{\xi}_8) f(\bar{\xi}_8, \bar{\xi}_6) f(\bar{\xi}_9, \bar{\xi}_6) f(\bar{z}, \bar{\eta}_6)}{f(\bar{\xi}_9, \bar{\eta}_6) f(\bar{z}, \bar{\xi}_6) f(\bar{\xi}_8, \bar{\eta}_6) f(\bar{\eta}_2, \bar{y})} |0\rangle. \quad (5.12)$$

In the same manner as before we take the sum over partitions $\bar{\xi}_9 \Rightarrow \{\bar{\xi}_3, \bar{\xi}_7\}$ in the braces in (5.12):

$$\sum K_k^{(r)}(q^{-2} \bar{\eta}_4 | \bar{\xi}_3) K_{b-k}^{(r)}(q^{-2} \bar{z} | \bar{\xi}_7) \frac{f(\bar{\xi}_3, \bar{\xi}_7)}{f(\bar{\xi}_3, \bar{z})} \\ = (-q)^{b-k} \sum K_k^{(r)}(q^{-2} \bar{\eta}_4 | \bar{\xi}_3) K_{b-k}^{(l)}(\bar{\xi}_7 | \bar{z}) \frac{f(\bar{\xi}_3, \bar{\xi}_7)}{f(\bar{\xi}_9, \bar{z})} = K_b^{(r)}(\{q^{-2} \bar{z}, q^{-2} \bar{\eta}_4\} | \bar{\xi}_9). \quad (5.13)$$

Thus, we obtain

$$\begin{aligned}
T_{32}(\bar{z})S_1 &= (-q)^{a-k}\lambda_2(\bar{x})\lambda_2(\bar{y})\lambda_2(\bar{z})\sum r_1(\bar{\eta}_6)r_3(\bar{\xi}_9)\mathsf{K}_a^{(l)}(\bar{\xi}_6|\{q^2\bar{x}, q^2\bar{y}\})\mathsf{K}_a^{(l)}(\bar{\eta}_6|q^2\bar{\xi}_6) \\
&\quad \times \mathsf{K}_b^{(r)}(\{q^{-2}\bar{z}, q^{-2}\bar{\eta}_4\}|\bar{\xi}_9)\mathsf{K}_{a-k}^{(r)}(q^{-2}\bar{x}|\bar{\eta}_2)\mathsf{K}_k^{(r)}(q^{-2}\bar{y}|\bar{\eta}_4)\mathsf{K}_{b-k}^{(l)}(\bar{\xi}_8|q^2\bar{z}) \\
&\quad \times \frac{f(\bar{\eta}_2, \bar{\eta}_4)f(\bar{\eta}_2, \bar{\eta}_6)f(\bar{\eta}_4, \bar{\eta}_6)f(\bar{\xi}_9, \bar{\xi}_8)f(\bar{\xi}_8, \bar{\xi}_6)f(\bar{\xi}_9, \bar{\xi}_6)f(\bar{z}, \bar{\eta}_6)}{f(\bar{\xi}_9, \bar{\eta}_6)f(\bar{z}, \bar{\xi}_6)f(\bar{\xi}_8, \bar{\eta}_6)f(\bar{\eta}_2, \bar{y})}|0\rangle. \quad (5.14)
\end{aligned}$$

Finally, we relabel subsets in (5.14) as: $\bar{\eta}_6 \rightarrow \bar{\eta}_h$, $\bar{\eta}_2 \rightarrow \bar{\eta}_{hi}$, $\bar{\eta}_4 \rightarrow \bar{\eta}_{hii}$, $\bar{\xi}_9 \rightarrow \bar{\xi}_i$, $\bar{\xi}_6 \rightarrow \bar{\xi}_{ii}$, $\bar{\xi}_8 \rightarrow \bar{\xi}_{iii}$ and set $\bar{u} = \bar{u}^B$, $\bar{v} = \bar{v}^B$. Then we have

$$\begin{aligned}
T_{32}(\bar{z})T_{31}(\bar{y})T_{21}(\bar{x})\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) &= (-q)^{a-k}\lambda_2(\bar{x})\lambda_2(\bar{y})\lambda_2(\bar{z})\sum r_1(\bar{\eta}_h)r_3(\bar{\xi}_i)\mathsf{K}_a^{(l)}(\bar{\eta}_h|q^2\bar{\xi}_{ii}) \\
&\quad \times \mathsf{K}_a^{(l)}(\bar{\xi}_{ii}|\{q^2\bar{x}, q^2\bar{y}\})\mathsf{K}_b^{(r)}(\{q^{-2}\bar{z}, q^{-2}\bar{\eta}_{hii}\}|\bar{\xi}_i)\mathsf{K}_{a-k}^{(r)}(q^{-2}\bar{x}|\bar{\eta}_{hi})\mathsf{K}_k^{(r)}(q^{-2}\bar{y}|\bar{\eta}_{hii})\mathsf{K}_{b-k}^{(l)}(\bar{\xi}_{iii}|q^2\bar{z}) \\
&\quad \times \frac{f(\bar{\eta}_{hi}, \bar{\eta}_{hii})f(\bar{\eta}_{hi}, \bar{\eta}_h)f(\bar{\eta}_{hii}, \bar{\eta}_h)f(\bar{\xi}_i, \bar{\xi}_{iii})f(\bar{\xi}_{iii}, \bar{\xi}_{ii})f(\bar{\xi}_i, \bar{\xi}_{ii})f(\bar{z}, \bar{\eta}_h)}{f(\bar{\xi}_i, \bar{\eta}_h)f(\bar{z}, \bar{\xi}_{ii})f(\bar{\xi}_{iii}, \bar{\eta}_h)f(\bar{\eta}_{hi}, \bar{y})}|0\rangle. \quad (5.15)
\end{aligned}$$

Here

- $\{\bar{x}, \bar{y}, \bar{u}^B\} = \bar{\eta} \Rightarrow \{\bar{\eta}_h, \bar{\eta}_{hi}, \bar{\eta}_{hii}\};$
- $\{\bar{x}, \bar{y}, \bar{z}, \bar{v}^B\} = \bar{\xi} \Rightarrow \{\bar{\xi}_i, \bar{\xi}_{ii}, \bar{\xi}_{iii}\}.$

6 Scalar product

6.1 Scalar product in terms of the highest coefficients

The equation (5.15) allows us to obtain an expression for the scalar product of Bethe vectors (3.1) in terms of sums over partitions of Bethe parameters. For this we take a representation for the dual Bethe vector (4.1)

$$\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) = \sum \frac{(-q)^k \mathsf{K}_k^{(l)}(\bar{u}_i^C|q^2\bar{v}_i^C)}{\lambda_2(\bar{v}_{ii}^C)\lambda_2(\bar{u}^C)f(\bar{v}_{ii}^C, \bar{u}^C)} f(\bar{v}_{ii}^C, \bar{v}_i^C)f(\bar{u}_{ii}^C, \bar{u}_i^C) \langle 0|T_{32}(\bar{v}_{ii}^C)T_{31}(\bar{u}_i^C)T_{21}(\bar{u}_{ii}^C). \quad (6.1)$$

Recall that here the sum is taken with respect to partitions $\bar{u}^C \Rightarrow \{\bar{u}_i^C, \bar{u}_{ii}^C\}$ and $\bar{v}^C \Rightarrow \{\bar{v}_i^C, \bar{v}_{ii}^C\}$ with $\#\bar{u}_i^C = \#\bar{v}_i^C = k$ and $k = 0, 1, \dots, \min(a, b)$. Thus, in order to calculate (3.1) we should take (6.1) and combine it with (5.15) setting there $\bar{y} = \bar{u}_i^C$, $\bar{x} = \bar{u}_{ii}^C$, and $\bar{z} = \bar{v}_{ii}^C$. Then

$$\begin{aligned}
S_{a,b} &= (-q)^a \sum r_1(\bar{\eta}_h)r_3(\bar{\xi}_i)\mathsf{K}_a^{(l)}(\bar{\xi}_{ii}|q^2\bar{u}^C)\mathsf{K}_a^{(l)}(\bar{\eta}_h|q^2\bar{\xi}_{ii})\mathsf{K}_b^{(r)}(\{q^{-2}\bar{v}_{ii}^C, q^{-2}\bar{\eta}_{hii}\}|\bar{\xi}_i) \\
&\quad \times \mathsf{K}_{b-k}^{(l)}(\bar{\xi}_{iii}|q^2\bar{v}_{ii}^C) \left\{ \mathsf{K}_{a-k}^{(r)}(q^{-2}\bar{u}_{ii}^C|\bar{\eta}_{hi})\mathsf{K}_k^{(r)}(q^{-2}\bar{u}_i^C|\bar{\eta}_{hii})\mathsf{K}_k^{(l)}(\bar{u}_i^C|q^2\bar{v}_i^C) \frac{f(\bar{u}_{ii}^C, \bar{u}_i^C)}{f(\bar{\eta}_{hi}, \bar{u}_i^C)} \right\} \\
&\quad \times \frac{f(\bar{\eta}_{hi}, \bar{\eta}_{hii})f(\bar{\eta}_{hi}, \bar{\eta}_h)f(\bar{\eta}_{hii}, \bar{\eta}_h)f(\bar{\xi}_i, \bar{\xi}_{iii})f(\bar{\xi}_{iii}, \bar{\xi}_{ii})f(\bar{\xi}_i, \bar{\xi}_{ii})f(\bar{v}_{ii}^C, \bar{\eta}_h)f(\bar{v}_{ii}^C, \bar{v}_i^C)}{f(\bar{\xi}_i, \bar{\eta}_h)f(\bar{v}_{ii}^C, \bar{\xi}_{ii})f(\bar{\xi}_{iii}, \bar{\eta}_h)f(\bar{v}_{ii}^C, \bar{u}^C)}. \quad (6.2)
\end{aligned}$$

Recall that here $\{\bar{u}^C, \bar{u}^B\} = \bar{\eta} \Rightarrow \{\bar{\eta}_i, \bar{\eta}_{ii}, \bar{\eta}_{iii}\}$ and $\{\bar{u}^C, \bar{v}_{ii}^C, \bar{v}^B\} = \bar{\xi} \Rightarrow \{\bar{\xi}_i, \bar{\xi}_{ii}, \bar{\xi}_{iii}\}$. For shortness we have also omitted the arguments of $S_{a,b}(\bar{u}^C; \bar{v}^C | \bar{u}^B; \bar{v}^B)$ in the l.h.s. of (6.2).

Now the expression (6.2) can be simplified. In particular, one can apply (A.4) and (B.1) to the terms in braces in order to take the sum over partitions $\bar{u}^C \Rightarrow \{\bar{u}_i^C, \bar{u}_{ii}^C\}$:

$$\begin{aligned} & \sum \mathbf{K}_k^{(r)}(q^{-2}\bar{u}_i^C | \bar{\eta}_{iii}) \mathbf{K}_k^{(l)}(\bar{u}_i^C | q^2\bar{v}_i^C) \mathbf{K}_{a-k}^{(r)}(q^{-2}\bar{u}_{ii}^C | \bar{\eta}_{ii}) \frac{f(\bar{u}_{ii}^C, \bar{u}_i^C)}{f(\bar{\eta}_{ii}, \bar{u}_i^C)} \\ &= (-q)^{a-k} \sum \mathbf{K}_k^{(r)}(\bar{u}_i^C | q^2\bar{\eta}_{iii}) \mathbf{K}_k^{(l)}(\bar{u}_i^C | q^2\bar{v}_i^C) \mathbf{K}_{a-k}^{(l)}(\bar{\eta}_{ii} | \bar{u}_{ii}^C) \frac{f(\bar{u}_{ii}^C, \bar{u}_i^C)}{f(\bar{\eta}_{ii}, \bar{u}_i^C)} \\ &= \frac{(-q)^a \mathbf{Z}_{a,k}^{(l)}(\bar{u}^C; \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\} | \bar{v}_i^C; q^{-2}\bar{\eta}_{iii})}{f(\bar{\eta}_{ii}, \bar{u}^C) f(\bar{\eta}_{iii}, \bar{u}^C) f(\bar{v}_i^C, \bar{u}^C)}. \end{aligned} \quad (6.3)$$

Then we obtain

$$\begin{aligned} S_{a,b} &= \frac{(-q)^{2a}}{f(\bar{v}^C, \bar{u}^C)} \sum r_1(\bar{\eta}_i) r_3(\bar{\xi}_i) \mathbf{Z}_{a,k}^{(l)}(\bar{u}^C; \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\} | \bar{v}_i^C; q^{-2}\bar{\eta}_{iii}) \\ &\times \mathbf{K}_b^{(r)}(\{q^{-2}\bar{v}_{ii}^C, q^{-2}\bar{\eta}_{iii}\} | \bar{\xi}_i) \left\{ \mathbf{K}_a^{(l)}(\bar{\xi}_{ii} | q^2\bar{u}^C) \mathbf{K}_a^{(l)}(\bar{\eta}_i | q^2\bar{\xi}_{ii}) \mathbf{K}_{b-k}^{(l)}(\bar{\xi}_{iii} | q^2\bar{v}_{ii}^C) \frac{f(\bar{\xi}_{iii}, \bar{\xi}_{ii})}{f(\bar{v}_{ii}^C, \bar{\xi}_{ii}) f(\bar{\xi}_{iii}, \bar{\eta}_i)} \right\} \\ &\times \frac{f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{\eta}_{ii}, \bar{\eta}_i) f(\bar{\eta}_{iii}, \bar{\eta}_i) f(\bar{\xi}_i, \bar{\xi}_{iii}) f(\bar{\xi}_i, \bar{\xi}_{ii}) f(\bar{v}_{ii}^C, \bar{\eta}_i) f(\bar{v}_{ii}^C, \bar{v}_i^C)}{f(\bar{\xi}_i, \bar{\eta}_i) f(\bar{\eta}_{ii}, \bar{u}^C) f(\bar{\eta}_{iii}, \bar{u}^C)}. \end{aligned} \quad (6.4)$$

Similarly we can take the sum of the terms in braces of (6.4) over partitions $\bar{\xi}_{ii}$ and $\bar{\xi}_{iii}$. Indeed, due to (A.4) and (B.1) we have

$$\begin{aligned} & \sum \mathbf{K}_a^{(l)}(\bar{\xi}_{ii} | q^2\bar{u}^C) \mathbf{K}_a^{(l)}(\bar{\eta}_i | q^2\bar{\xi}_{ii}) \mathbf{K}_{b-k}^{(l)}(\bar{\xi}_{iii} | q^2\bar{v}_{ii}^C) \frac{f(\bar{\xi}_{iii}, \bar{\xi}_{ii})}{f(\bar{v}_{ii}^C, \bar{\xi}_{ii}) f(\bar{\xi}_{iii}, \bar{\eta}_i)} \\ &= (-q)^{k-b-a} \sum \mathbf{K}_a^{(l)}(\bar{\xi}_{ii} | q^2\bar{u}^C) \mathbf{K}_a^{(r)}(\bar{\xi}_{ii} | \bar{\eta}_i) \mathbf{K}_{b-k}^{(r)}(\bar{v}_{ii}^C | \bar{\xi}_{iii}) \frac{f(\bar{\xi}_{iii}, \bar{\xi}_{ii})}{f(\bar{v}_{ii}^C, \bar{\xi}_0) f(\bar{\xi}_0, \bar{\eta}_i)} \\ &= (-q)^{k-b-2a} \frac{\mathbf{Z}_{a+b-k,a}^{(r)}(\bar{\xi}_0; \{\bar{v}_{ii}^C, \bar{u}^C\} | q^{-2}\bar{\eta}_i; q^{-2}\bar{u}^C)}{f(\bar{v}_{ii}^C, \bar{\xi}_0) f(\bar{u}^C, \bar{\xi}_0)}, \end{aligned} \quad (6.5)$$

where $\bar{\xi}_0 = \{\bar{\xi}_{ii}, \bar{\xi}_{iii}\}$. Thus, we obtain

$$\begin{aligned} S_{a,b} &= \sum \frac{(-q)^{k-b}}{f(\bar{v}^C, \bar{u}^C)} r_1(\bar{\eta}_i) r_3(\bar{\xi}_i) \mathbf{K}_b^{(r)}(\{q^{-2}\bar{v}_{ii}^C, q^{-2}\bar{\eta}_{iii}\} | \bar{\xi}_i) \mathbf{Z}_{a,k}^{(l)}(\bar{u}^C; \bar{\eta}_0 | \bar{v}_i^C; q^{-2}\bar{\eta}_{iii}) \\ &\times \mathbf{Z}_{a+b-k,a}^{(r)}(\bar{\xi}_0; \{\bar{v}_{ii}^C, \bar{u}^C\} | q^{-2}\bar{\eta}_i; q^{-2}\bar{u}^C) \frac{f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{\eta}_0, \bar{\eta}_i) f(\bar{\xi}_i, \bar{\xi}_0) f(\bar{v}_{ii}^C, \bar{\eta}_i) f(\bar{v}_{ii}^C, \bar{v}_i^C)}{f(\bar{\xi}_i, \bar{\eta}_i) f(\bar{\eta}_0, \bar{u}^C) f(\bar{v}_{ii}^C, \bar{\xi}_0) f(\bar{u}^C, \bar{\xi}_0)}, \end{aligned} \quad (6.6)$$

where $\bar{\eta}_0 = \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\}$.

Remark. Strictly speaking, one should understand the sets $\bar{\eta}$ and $\bar{\xi}$ in the equation (6.6) as

$$\begin{aligned} \bar{\eta} &= \{\bar{u}^B + \epsilon, \bar{u}^C + \epsilon\}, \\ \bar{\xi} &= \{\bar{u}^C + \epsilon, \bar{v}^B + \epsilon, \bar{v}_{ii}^C + \epsilon\}, \end{aligned} \quad \text{at } \epsilon \rightarrow 0. \quad (6.7)$$

The matter is that individual factors in (6.6) may have singularities, if we set $\epsilon = 0$. For instance, the highest coefficient $Z_{a,k}^{(l)}(\bar{u}^C; \bar{\eta}_0 | \bar{v}_{ii}^C; q^{-2} \bar{\eta}_{iii})$ is singular if $\bar{\eta}_0 \cap \bar{u}^C \neq \emptyset$ (see (B.3)). These singularities, of course, are compensated by the product $f^{-1}(\bar{\eta}_0, \bar{u}^C)$, but for evaluating the limit we should have $\epsilon \neq 0$. In order to lighten the formulas we do not write this auxiliary parameter ϵ explicitly, but one has to keep it in mind when doing the calculations.

The equation (6.6) already gives the representation for the scalar product of Bethe vectors in term of the sum over partitions of Bethe parameters. However it is not convenient for further applications. In particular, it contains many terms which actually cancel each other. Below we simplify (6.6) making several reductions of the highest coefficients.

6.2 The first reduction of the highest coefficients

The first simplification of (6.6) is based on the following

Proposition 6.1. *The subset $\bar{\xi}_i$ in (6.6) does not contain the elements from the set \bar{u}^C , that is $\bar{\xi}_i \cap \bar{u}^C = \emptyset$.*

Proof. The representation for the scalar product (6.6) can be written in the following form

$$S_{a,b} \equiv \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \sum r_1(\bar{\eta}_i) r_3(\bar{\xi}_i) W_{\text{part}}(\bar{\eta}; \bar{\xi}; q), \quad (6.8)$$

where $W_{\text{part}}(\bar{\eta}; \bar{\xi}; q)$ is a rational function depending on the partitions. For the moment its explicit form is not important, however we have shown explicitly that W_{part} depends on the parameter q . The sum is taken over partitions of the set $\bar{\eta} = \{\bar{u}^B, \bar{u}^C\}$ and of the set $\bar{\xi} = \{\bar{u}^C, \bar{v}^B, \bar{v}_{ii}^C\}$.

We can apply the isomorphism φ to the equation (6.8). Due to (4.10) in the r.h.s. one should simply make the replacement $r_1 \leftrightarrow r_3$:

$$\varphi(S_{a,b}) = \sum r_3(\bar{\eta}_i) r_1(\bar{\xi}_i) W_{\text{part}}(\bar{\eta}; \bar{\xi}; q). \quad (6.9)$$

On the other hand due to (4.9) we have in the l.h.s.

$$\varphi(S_{a,b}) = \varphi(\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)) \varphi(\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)) = \mathbb{C}_{q^{-1}}^{b,a}(\bar{v}^C; \bar{u}^C) \mathbb{B}_{q^{-1}}^{b,a}(\bar{v}^B; \bar{u}^B). \quad (6.10)$$

Calculating the scalar product in (6.10) via (6.8) we obtain

$$\varphi(S_{a,b}) = \sum r_1(\bar{\eta}'_i) r_3(\bar{\xi}'_i) W_{\text{part}}(\bar{\eta}'; \bar{\xi}'; q^{-1}), \quad (6.11)$$

where now the sum is taken over partitions of the set $\bar{\eta}' = \{\bar{v}^B, \bar{v}^C\}$ and the set $\bar{\xi}' = \{\bar{v}^C, \bar{u}^B, \bar{u}_{ii}^C\}$. Thus, we arrive at

$$\sum r_1(\bar{\eta}'_i) r_3(\bar{\xi}'_i) W_{\text{part}}(\bar{\eta}'; \bar{\xi}'; q^{-1}) = \sum r_3(\bar{\eta}_i) r_1(\bar{\xi}_i) W_{\text{part}}(\bar{\eta}; \bar{\xi}; q). \quad (6.12)$$

Since r_1 and r_3 are free functional parameters we conclude that for an arbitrary subset $\bar{\eta}'_i \subset \bar{\eta}'$ there exists a subset $\bar{\xi}_i \subset \bar{\xi}$ such that $\bar{\eta}'_i = \bar{\xi}_i$ and vice versa. But $\bar{\eta}'_i \subset \{\bar{v}^B, \bar{v}^C\}$, hence, $\bar{\xi}_i \subset \{\bar{v}^B, \bar{v}^C\}$, and thus $\bar{\xi}_i \cap \bar{u}^C = \emptyset$. \square

Due to Proposition 6.1 if $\bar{\xi}_i \cap \bar{u}^C \neq \emptyset$, then the corresponding term in the sum (6.6) vanishes. Hence, $\bar{u}^C \subset \bar{\xi}_0$, and we can set $\bar{\xi}_0 = \{\bar{\xi}_{ii}, \bar{u}^C\}$ and $\{\bar{\xi}_i, \bar{\xi}_{ii}\} = \bar{\xi} = \{\bar{v}^B, \bar{v}_{ii}^C\}$. Then we obtain a possibility to simplify the highest coefficient $Z_{a+b-k,a}^{(r)}$ via (B.3) and (B.5)

$$\begin{aligned} \frac{1}{f(\bar{\xi}_0, \bar{u}^C)} Z_{a+b-k,a}^{(r)}(\{\bar{\xi}_{ii}, \bar{u}^C\}; \{\bar{v}_{ii}^C, \bar{u}^C\} | q^{-2}\bar{\eta}_i; q^{-2}\bar{u}^C) &= \frac{f(\bar{v}_{ii}^C, \bar{u}^C)}{f(\bar{u}^C, \bar{\eta}_i)} Z_{b-k,a}^{(r)}(\bar{\xi}_{ii}; \bar{v}_{ii}^C | q^{-2}\bar{\eta}_i; q^{-2}\bar{u}^C) \\ &= \frac{Z_{a,b-k}^{(r)}(\bar{\eta}_i; \bar{u}^C | \bar{\xi}_{ii}; \bar{v}_{ii}^C)}{f(\bar{u}^C, \bar{\eta}_i) f(\bar{\xi}_{ii}, \bar{\eta}_i)}. \end{aligned} \quad (6.13)$$

Thus, the equation (6.6) turns into

$$\begin{aligned} S_{a,b} &= \sum \frac{(-q)^{k-b}}{f(\bar{v}^C, \bar{u}^C)} r_1(\bar{\eta}_i) r_3(\bar{\xi}_i) K_b^{(r)}(\{q^{-2}\bar{v}_{ii}^C, q^{-2}\bar{\eta}_{iii}\} | \bar{\xi}_i) Z_{a,k}^{(l)}(\bar{u}^C; \bar{\eta}_0 | \bar{v}_i^C; q^{-2}\bar{\eta}_{iii}) \\ &\quad \times Z_{a,b-k}^{(r)}(\bar{\eta}_i; \bar{u}^C | \bar{\xi}_{ii}; \bar{v}_{ii}^C) \frac{f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{\eta}_0, \bar{\eta}_i) f(\bar{\xi}_i, \bar{\xi}_{ii}) f(\bar{\xi}_i, \bar{u}^C) f(\bar{v}_{ii}^C, \bar{v}_i^C)}{f(\bar{v}^B, \bar{\eta}_i) f(\bar{\eta}_0, \bar{u}^C) f(\bar{v}_{ii}^C, \bar{u}^C) f(\bar{v}_{ii}^C, \bar{\xi}_{ii}) f(\bar{u}^C, \bar{\eta}_i)}, \end{aligned} \quad (6.14)$$

where we have used $\{\bar{\xi}_i, \bar{\xi}_{ii}\} = \{\bar{v}^B, \bar{v}_{ii}^C\}$.

6.3 The second reduction

For further reductions we should specify the subsets $\bar{\eta}_k$ and $\bar{\xi}_k$ in terms of the original variables. Recall that the sum in (6.14) is taken over partitions:

- $\{\bar{v}^B, \bar{v}_{ii}^C\} \Rightarrow \{\bar{\xi}_i, \bar{\xi}_{ii}\}$;
- $\{\bar{u}^C, \bar{u}^B\} \Rightarrow \{\bar{\eta}_i, \bar{\eta}_0\}$ and then $\bar{\eta}_0 \Rightarrow \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\}$.

We set

$$\begin{aligned} \bar{\eta}_i &= \{\bar{u}_i^B, \bar{u}_{ii}^C\}, \\ \bar{\eta}_0 &= \{\bar{u}_{ii}^B, \bar{u}_i^C\}, \end{aligned} \quad (6.15)$$

with $\#\bar{u}_i^C = \#\bar{u}_{ii}^B = n$, $n = 0, 1, \dots, a$. Note that we do not specify subsets $\bar{\eta}_{ii}$ and $\bar{\eta}_{iii}$, however $\{\bar{\eta}_{ii}, \bar{\eta}_{iii}\} = \{\bar{u}_{ii}^B, \bar{u}_i^C\}$.

The partitions of the sets \bar{v}^B and \bar{v}^C are more sophisticated.

We start with \bar{v}^C , that appears in (6.14) as $\bar{v}^C \Rightarrow \{\bar{v}_i^C, \bar{v}_{ii}^C\}$ with $k = \#\bar{v}_i^C$. First of all we divide the set \bar{v}_{ii}^C into additional subsets $\bar{v}_{ii}^C \Rightarrow \{\bar{v}_{ii}^C, \bar{v}_0^C\}$ with $\#\bar{v}_{ii}^C = b - m$, $m = 0, 1, \dots, b - k$. Then we define the subset \bar{v}_i^C such that $\{\bar{v}_i^C, \bar{v}_{ii}^C\} = \bar{v}^C$. Evidently $\#\bar{v}_i^C = m$ and $\bar{v}_i^C = \{\bar{v}_i^C, \bar{v}_0^C\}$.

Finally we divide the set \bar{v}^B as $\bar{v}^B \Rightarrow \{\bar{v}_i^B, \bar{v}_{ii}^B\}$ with $\#\bar{v}_i^B = m$ and set

$$\begin{aligned} \bar{\xi}_i &= \{\bar{v}_i^B, \bar{v}_{ii}^C\}, \\ \bar{\xi}_{ii} &= \{\bar{v}_{ii}^B, \bar{v}_0^C\}. \end{aligned} \quad (6.16)$$

Hereby

$$\begin{aligned} \bar{v}_{ii}^C &= \{\bar{v}_{ii}^C, \bar{v}_0^C\}, & \#\bar{v}_i^C &= \#\bar{v}_i^B = m, & \#\bar{v}_i^C &= k, \\ \bar{v}_i^C &= \bar{v}_i^C \setminus \bar{v}_0^C, & \#\bar{v}_{ii}^C &= \#\bar{v}_{ii}^B = b - m, & \#\bar{v}_0^C &= m - k. \end{aligned} \quad (6.17)$$

Now we can apply (B.3) to the highest coefficient $Z_{a,k}^{(l)}$:

$$\frac{1}{f(\bar{\eta}_0, \bar{u}^C)} Z_{a,k}^{(l)}(\bar{u}^C; \bar{\eta}_0 | \bar{v}_1^C; q^{-2} \bar{\eta}_{iii}) = \frac{f(\bar{v}_1^C, \bar{u}_1^C)}{f(\bar{u}_\Pi^B, \bar{u}_\Pi^C)} Z_{a-n,k}^{(l)}(\bar{u}_\Pi^C; \bar{u}_\Pi^B | \bar{v}_1^C; q^{-2} \bar{\eta}_{iii}). \quad (6.18)$$

For simplification of $Z_{a,b-k}^{(r)}$ we use successively (B.3) and then (B.4)

$$\frac{Z_{a,b-k}^{(r)}(\bar{\eta}_1; \bar{u}^C | \bar{\xi}_{ii}; \bar{v}_{ii}^C)}{f(\bar{u}^C, \bar{\eta}_1) f(\bar{v}_{ii}^C, \bar{\xi}_{ii})} = \frac{f(\bar{v}_\Pi^B, \bar{u}_\Pi^C) f(\bar{v}_0^C, \bar{u}^C)}{f(\bar{u}_1^C, \bar{u}_1^B) f(\bar{v}_\Pi^C, \bar{v}_\Pi^B)} Z_{n,b-m}^{(r)}(\bar{u}_1^B; \bar{u}_1^C | \bar{v}_\Pi^B; \bar{v}_\Pi^C). \quad (6.19)$$

Finally, due to (A.3) we have

$$\mathbf{K}_b^{(r)}(\{q^{-2} \bar{v}_{ii}^C, q^{-2} \bar{\eta}_{iii}\} | \bar{\xi}_i) = (-q)^{b-m} \mathbf{K}_m^{(r)}(\{q^{-2} \bar{v}_0^C, q^{-2} \bar{\eta}_{iii}\} | \bar{v}_1^B). \quad (6.20)$$

Now we should substitute (6.18)–(6.20) into (6.14). After some elementary but rather exhausting algebra we obtain

$$\begin{aligned} S_{a,b} &= f^{-1}(\bar{v}^C, \bar{u}^C) f^{-1}(\bar{v}^B, \bar{u}^B) \sum r_1(\bar{u}_1^B) r_1(\bar{u}_\Pi^C) r_3(\bar{v}_1^B) r_3(\bar{v}_\Pi^C) Z_{n,b-m}^{(r)}(\bar{u}_1^B; \bar{u}_1^C | \bar{v}_\Pi^B; \bar{v}_\Pi^C) \\ &\times (-q)^{k-m} \mathbf{K}_m^{(r)}(q^{-2} \bar{v}_0^C, q^{-2} \bar{\eta}_{iii} | \bar{v}_1^B) Z_{a-n,k}^{(l)}(\bar{u}_\Pi^C; \bar{u}_\Pi^B | \bar{v}_1^C; q^{-2} \bar{\eta}_{iii}) f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{v}_0^C, \bar{v}_1^C) f(\bar{v}_1^B, \bar{v}_0^C) f(\bar{v}_1^C, \bar{u}_1^C) \\ &\times f(\bar{u}_\Pi^B, \bar{u}_1^B) f(\bar{u}_1^C, \bar{u}_\Pi^C) f(\bar{v}_1^B, \bar{v}_\Pi^B) f(\bar{v}_\Pi^C, \bar{v}_1^C) f(\bar{v}_1^B, \bar{u}_1^C) f(\bar{v}^B, \bar{u}^B). \end{aligned} \quad (6.21)$$

Recall that here $\{\bar{\eta}_{ii}, \bar{\eta}_{iii}\} = \{\bar{u}_\Pi^B, \bar{u}_1^C\}$.

Thus, the representation for the scalar product $S_{a,b}$ has the form (3.2), where the function W_{part} has the following representation:

$$\begin{aligned} W_{\text{part}} \left(\begin{array}{cc} \bar{u}_\Pi^C, \bar{u}_\Pi^B & \bar{u}_1^C, \bar{u}_1^B \\ \bar{v}_1^C, \bar{v}_1^B & \bar{v}_\Pi^C, \bar{v}_\Pi^B \end{array} \right) &= Z_{n,b-m}^{(r)}(\bar{u}_1^B; \bar{u}_1^C | \bar{v}_\Pi^B; \bar{v}_\Pi^C) \\ &\times f(\bar{u}_\Pi^B, \bar{u}_1^B) f(\bar{u}_1^C, \bar{u}_\Pi^C) f(\bar{v}_1^B, \bar{v}_\Pi^B) f(\bar{v}_\Pi^C, \bar{v}_1^C) f(\bar{v}_1^B, \bar{u}_1^C) f(\bar{v}_1^C, \bar{u}_1^C) f(\bar{v}^B, \bar{u}^B) \widetilde{W}. \end{aligned} \quad (6.22)$$

The factor \widetilde{W} is equal to the sum over additional partitions $\bar{v}_1^C \Rightarrow \{\bar{v}_0^C, \bar{v}_1^C\}$ and $\{\bar{u}_\Pi^B, \bar{u}_1^C\} \Rightarrow \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\}$:

$$\begin{aligned} \widetilde{W} &= \sum (-q)^{k-m} \mathbf{K}_m^{(r)}(q^{-2} \bar{v}_0^C, q^{-2} \bar{\eta}_{iii} | \bar{v}_1^B) Z_{a-n,k}^{(l)}(\bar{u}_\Pi^C; \bar{u}_\Pi^B | \bar{v}_1^C; q^{-2} \bar{\eta}_{iii}) \\ &\times f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{v}_0^C, \bar{v}_1^C) f(\bar{v}_1^B, \bar{v}_0^C) f^{-1}(\bar{v}_0^C, \bar{u}_1^C). \end{aligned} \quad (6.23)$$

This sum can be explicitly calculated due to the following identity.

Proposition 6.2. *Let a, b, n be non-negative integers and $\bar{t}, \bar{x}, \bar{s}, \bar{y}, \bar{z}$ be five sets of generic complex variables with cardinalities $\#\bar{t} = \#\bar{x} = a$, $\#\bar{s} = \#\bar{y} = b$, and $\#\bar{z} = n$. Then*

$$\begin{aligned} f^{-1}(\bar{y}, \bar{\eta}) Z_{a,b}^{(l,r)}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) &= \sum (-q)^{\mp \ell} \mathbf{K}_b^{(r,l)}(\{\bar{s}_1 q^{-2}, \bar{\eta}_1 q^{-2}\} | \bar{y}) Z_{a,b-k}^{(l,r)}(\bar{t}; \bar{x} | \bar{s}_\Pi; \bar{\eta}_1 q^{-2}) \\ &\times f(\bar{s}_1, \bar{s}_\Pi) f(\bar{\eta}_\Pi, \bar{\eta}_1) f(\bar{y}, \bar{s}_1) f^{-1}(\bar{s}_1, \bar{z}). \end{aligned} \quad (6.24)$$

Here $\bar{\eta}$ is a union of two sets: $\bar{\eta} = \{\bar{x}, \bar{z}\}$. The sum is taken over partitions of the set $\bar{s} \Rightarrow \{\bar{s}_1, \bar{s}_\Pi\}$ with $\#\bar{s}_1 = \ell \in [0, \dots, b]$ and the set $\bar{\eta} \Rightarrow \{\bar{\eta}_1, \bar{\eta}_\Pi\}$ with $\#\bar{\eta}_1 = b - \ell$.

This identity was proved in [5]. We give another proof in appendix C. It is easy to see that making in (6.24) the following change of variables:

- $\bar{\eta}_I \rightarrow \bar{\eta}_{III}$ and $\bar{\eta}_{II} \rightarrow \bar{\eta}_{III}$;
- $\bar{s}_I \rightarrow \bar{v}_0^C$ and $\bar{s}_{II} \rightarrow \bar{v}_1^C$;
- $\bar{y} = \bar{v}_1^B$, $\bar{t} = \bar{u}_{II}^C$, $\bar{x} = \bar{u}_{II}^B$, $\bar{z} = \bar{u}_I^C$;
- $a \rightarrow a - n$, $\ell \rightarrow m - k$, $b \rightarrow m$.

we reproduce the sum over partitions in (6.23), what gives us

$$\widetilde{W} = \frac{Z_{a-n,m}^{(l)}(\bar{u}_{II}^C; \bar{u}_{II}^B | \bar{v}_I^C; \bar{v}_I^B)}{f(\bar{v}_I^B, \bar{u}_I^C) f(\bar{v}_I^C, \bar{u}_{II}^B)}. \quad (6.25)$$

Substituting this into (6.21) we reproduce (3.3). Thus, we have arrived at the trigonometric analog of the formula for the scalar product of Bethe vectors obtained in [26] for the models with $GL(3)$ -invariant R -matrix.

Conclusion

The main goal of this paper was to prove the representation (3.2), (3.3) for the scalar product of Bethe vectors in the models with $GL(3)$ trigonometric R -matrix. As we have mentioned already, this representation looks very similar to the one obtained in [26]. The main difference is that the rational coefficients W_{part} in (3.3) depend now on two different highest coefficients, while in the models with $GL(3)$ -invariant R -matrix there exist only one highest coefficient.

As we have mentioned in the Introduction, the representation (3.2) is not convenient for direct applications to the study of correlation functions and form factors of local operators. It is worth mentioning however, that this representation is obtained for the most general case, when the Bethe parameters of both vectors are arbitrary complex numbers. Moreover, we considered the vacuum eigenvalues $\lambda_i(u)$ of the operators $T_{ii}(u)$ as free functional parameters. In this case one hardly can hope to simplify the expression for the scalar product. On the other hand, calculating the correlation functions and form factors in concrete quantum models, we always deal with a specific representation of \mathcal{A}_q algebra. This fixes the functions $\lambda_i(u)$. Furthermore, in the case of correlation functions we have to consider particular cases of scalar products, where most of the Bethe parameters satisfy Bethe equations. This gives us a possibility to calculate at least a part of the sums over partitions in equation (3.2). In this way one can obtain compact determinant representations for some scalar products and form factors of the monodromy matrix entries. This was done already for the scalar products in the models with $GL(3)$ -invariant R -matrix (see [28], [29]). One can expect that similar results can be obtained starting from the representation (3.2), (3.3).

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A Properties of Izergin determinants

Most of the properties of the left and right Izergin determinants easily follow directly from their definitions. We give below a list of these properties. We remind that the superscript (l, r) on K means that the equality is valid for $K^{(l)}$ and for $K^{(r)}$ with appropriate choice of component (first/up or second/down) throughout the equality.

Initial condition:

$$K_1^{(l)}(\bar{x}|\bar{y}) = x g(x, y), \quad K_1^{(r)}(\bar{x}|\bar{y}) = y g(x, y). \quad (\text{A.1})$$

Scaling:

$$K_n^{(l, r)}(\alpha \bar{x} | \alpha \bar{y}) = K_n^{(l, r)}(\bar{x} | \bar{y}). \quad (\text{A.2})$$

Reduction:

$$K_{n+1}^{(l, r)}(\{\bar{x}, q^{-2}z\} | \{\bar{y}, z\}) = K_{n+1}^{(l, r)}(\{\bar{x}, z\} | \{\bar{y}, q^2z\}) = -q^{\mp 1} K_n^{(l, r)}(\bar{x} | \bar{y}). \quad (\text{A.3})$$

Inverse order of arguments:

$$K_n^{(l, r)}(q^{-2}\bar{x} | \bar{y}) = (-q)^{\mp n} f^{-1}(\bar{y}, \bar{x}) K_n^{(r, l)}(\bar{y} | \bar{x}), \quad (\text{A.4})$$

$$K_{n; q^{-1}}^{(l, r)}(\bar{x} | \bar{y}) = K_{n; q}^{(r, l)}(\bar{y} | \bar{x}), \quad (\text{A.5})$$

where $K_{n; q^{-1}}^{(l, r)}$ means $K_n^{(l, r)}$ with q replaced by q^{-1} . We have put an additional index q^{-1} or q in (A.5) to stress this replacement.

Residues in the poles:

$$K_{n+1}^{(l, r)}(\{\bar{x}, z\} | \{\bar{y}, z'\}) \Big|_{z' \rightarrow z} = f(z, z') f(z, \bar{y}) f(\bar{x}, z) K_n^{(l, r)}(\bar{x} | \bar{y}) + \text{reg}, \quad (\text{A.6})$$

where reg means the regular part.

The Izergin determinants satisfy also summation identities.

Lemma A.1. *Let $\bar{\gamma}$, $\bar{\alpha}$ and $\bar{\beta}$ be three sets of complex variables with $\#\alpha = m_1$, $\#\beta = m_2$, and $\#\gamma = m_1 + m_2$. Then*

$$\sum K_{m_1}^{(l, r)}(\bar{\gamma}_I | \bar{\alpha}) K_{m_2}^{(r, l)}(\bar{\beta} | \bar{\gamma}_{II}) f(\bar{\gamma}_{II}, \bar{\gamma}_I) = (-q)^{\mp m_1} f(\bar{\gamma}, \bar{\alpha}) K_{m_1+m_2}^{(r, l)}(\{\bar{\alpha} q^{-2}, \bar{\beta}\} | \bar{\gamma}). \quad (\text{A.7})$$

The sum is taken with respect to all partitions of the set $\bar{\gamma} \Rightarrow \{\bar{\gamma}_I, \bar{\gamma}_{II}\}$ with $\#\bar{\gamma}_I = m_1$ and $\#\bar{\gamma}_{II} = m_2$. Due to (A.4) the equation (A.7) can be also written in the form

$$\sum K_{m_1}^{(l, r)}(\bar{\gamma}_I | \bar{\alpha}) K_{m_2}^{(r, l)}(\bar{\beta} | \bar{\gamma}_{II}) f(\bar{\gamma}_{II}, \bar{\gamma}_I) = (-q)^{\pm m_2} f(\bar{\beta}, \bar{\gamma}) K_{m_1+m_2}^{(l, r)}(\bar{\gamma} | \{\bar{\alpha}, \bar{\beta} q^2\}). \quad (\text{A.8})$$

This statement is a simple corollary of Lemma 1 of the work [28].

B Properties of $Z_{a,b}$

The proofs of the properties listed below are given in [5].

B.1 Sum formulas

Let $a \geq b$. Then

$$\sum \kappa_b^{(r,l)}(\bar{t}_I | \bar{y} q^2) \kappa_b^{(l,r)}(\bar{t}_I | \bar{s} q^2) \kappa_{a-b}^{(l,r)}(\bar{\xi} | \bar{t}_{II}) f(\bar{t}_{II}, \bar{t}_I) = (-q)^{\pm b} \frac{Z_{a,b}^{(l,r)}(\bar{t}; \{\bar{\xi}, \bar{y}\} | \bar{s}; \bar{y} q^{-2})}{f(\bar{y}, \bar{t}) f(\bar{s}, \bar{t})}, \quad (\text{B.1})$$

$$\sum \kappa_b^{(l,r)}(\bar{t}_I | \bar{y} q^2) \kappa_b^{(r,l)}(\bar{t}_I | \bar{s} q^2) \kappa_{a-b}^{(l,r)}(\bar{\xi} | \bar{t}_{II}) f(\bar{t}_{II}, \bar{t}_I) = (-q)^{\pm b} \frac{Z_{a,b}^{(l,r)}(\bar{t}; \{\bar{\xi}, \bar{s}\} | \bar{y}; \bar{s} q^{-2})}{f(\bar{y}, \bar{t}) f(\bar{s}, \bar{t})}. \quad (\text{B.2})$$

Here the sum is taken over partitions $\bar{t} \Rightarrow \{\bar{t}_I, \bar{t}_{II}\}$ with $\#\bar{t}_I = b$.

B.2 Reductions of the highest coefficients

Let $\#\bar{z} = n$. Then

$$\lim_{\bar{z}' \rightarrow \bar{z}} f^{-1}(\bar{z}', \bar{z}) Z_{a+n,b}^{(l,r)}(\{\bar{t}, \bar{z}\}; \{\bar{x}, \bar{z}'\} | \bar{s}; \bar{y}) = \bar{z}^{-1} f(\bar{z}, \bar{t}) f(\bar{x}, \bar{z}) f(\bar{s}, \bar{z}) Z_{a,b}^{(l,r)}(\bar{t}; \bar{x} | \bar{s}; \bar{y}). \quad (\text{B.3})$$

and

$$\lim_{\bar{z}' \rightarrow \bar{z}} f^{-1}(\bar{z}', \bar{z}) Z_{a,b+n}^{(l,r)}(\bar{t}; \bar{x} | \{\bar{s}, \bar{z}\}; \{\bar{y}, \bar{z}'\}) = \bar{z}^{-1} f(\bar{z}, \bar{x}) f(\bar{z}, \bar{s}) f(\bar{y}, \bar{z}) Z_{a,b}^{(l,r)}(\bar{t}; \bar{x} | \bar{s}; \bar{y}), \quad (\text{B.4})$$

B.3 Reverse order of arguments

$$Z_{b,a}^{(l,r)}(\bar{s}; \bar{y} | \bar{t} q^{-2}; \bar{x} q^{-2}) = f^{-1}(\bar{y}, \bar{x}) f^{-1}(\bar{s}, \bar{t}) Z_{a,b}^{(l,r)}(\bar{t}; \bar{x} | \bar{s}; \bar{y}). \quad (\text{B.5})$$

C Proof of Proposition 6.2

We start with the equation (6.14). Let us set there $\bar{\eta}_i = \bar{u}^C$ and $\bar{\xi}_i = \bar{v}^B$. This implies $\bar{\eta}_0 = \bar{u}^B$ and $\bar{\xi}_{ii} = \bar{v}_{ii}^C$. We know that the rational coefficient of the product $r_1(\bar{u}^C) r_3(\bar{v}^B)$ is the highest coefficient $Z^{(l)}$ (up to a normalization):

$$S_{a,b} = r_1(\bar{u}^C) r_3(\bar{v}^B) \frac{Z_{a,b}^{(l)}(\bar{u}^C; \bar{u}^B | \bar{v}^C; \bar{v}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} + \dots, \quad (\text{C.1})$$

where dots mean the terms corresponding to all other subsets $\bar{\eta}_i$ and $\bar{\xi}_i$. On the other hand, we still have the sum over partitions $\bar{v}^C \Rightarrow \{\bar{v}_i^C, \bar{v}_{ii}^C\}$ and $\bar{\eta}_0 = \bar{u}^B \Rightarrow \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\}$ in the equation

(6.14). Thus, we obtain

$$\begin{aligned}
r_1(\bar{u}^C) r_3(\bar{v}^B) \frac{Z_{a,b}^{(l)}(\bar{u}^C; \bar{u}^B | \bar{v}^C; \bar{v}^B)}{f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B)} &= \sum \frac{(-q)^{k-b}}{f(\bar{v}^C, \bar{u}^C)} r_1(\bar{u}^C) r_3(\bar{v}^B) \mathbf{K}_b^{(r)}(\{q^{-2}\bar{v}_{ii}^C, q^{-2}\bar{\eta}_{iii}\} | \bar{v}^B) \\
&\times Z_{a,k}^{(l)}(\bar{u}^C; \bar{u}^B | \bar{v}_i^C; q^{-2}\bar{\eta}_{iii}) f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{v}^B, \bar{v}_{ii}^C) f(\bar{v}_{ii}^C, \bar{v}_i^C) \\
&\times \lim_{\substack{\bar{\xi}_{ii} \rightarrow \bar{v}_{ii}^C \\ \bar{\eta}_i \rightarrow \bar{u}^C}} \frac{Z_{a,b-k}^{(r)}(\bar{\eta}_i; \bar{u}^C | \bar{\xi}_{ii}; \bar{v}_{ii}^C)}{f(\bar{v}_{ii}^C, \bar{u}^C) f(\bar{v}_{ii}^C, \bar{\xi}_{ii}) f(\bar{u}^C, \bar{\eta}_i)}, \quad (C.2)
\end{aligned}$$

where the sum is taken over partitions $\bar{v}^C \Rightarrow \{\bar{v}_i^C, \bar{v}_{ii}^C\}$ and $\bar{u}^B \Rightarrow \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\}$. Due to (B.4), (B.3) one has

$$\lim_{\substack{\bar{\xi}_{ii} \rightarrow \bar{v}_{ii}^C \\ \bar{\eta}_i \rightarrow \bar{u}^C}} \frac{Z_{a,b-k}^{(r)}(\bar{\eta}_i; \bar{u}^C | \bar{\xi}_{ii}; \bar{v}_{ii}^C)}{f(\bar{v}_{ii}^C, \bar{u}^C) f(\bar{v}_{ii}^C, \bar{\xi}_{ii}) f(\bar{u}^C, \bar{\eta}_i)} = 1, \quad (C.3)$$

and we arrive at

$$\begin{aligned}
f^{-1}(\bar{v}^B, \bar{u}^B) Z_{a,b}^{(l)}(\bar{u}^C; \bar{u}^B | \bar{v}^C; \bar{v}^B) &= \sum (-q)^{k-b} \mathbf{K}_b^{(r)}(\{q^{-2}\bar{v}_{ii}^C, q^{-2}\bar{\eta}_{iii}\} | \bar{v}^B) \\
&\times Z_{a,k}^{(l)}(\bar{u}^C; \bar{u}^B | \bar{v}_i^C; q^{-2}\bar{\eta}_{iii}) f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{v}^B, \bar{v}_{ii}^C) f(\bar{v}_{ii}^C, \bar{v}_i^C). \quad (C.4)
\end{aligned}$$

One can easily recognize in this equation a particular case of the identity (6.24) corresponding to $\bar{z} = \emptyset$.

In order to reproduce (6.24) with $\bar{z} \neq \emptyset$, we replace in (C.4) a by $a+n$ and set there

$$\begin{aligned}
\bar{u}^C &= \{\bar{t}, \bar{z}\}, & \bar{v}^C &= \bar{s}, \\
\bar{u}^B &= \{\bar{x}, \bar{z}'\}, & \bar{v}^B &= \bar{y},
\end{aligned} \quad (C.5)$$

with $\#\bar{t} = \#\bar{x} = a$ and $\#\bar{z} = \#\bar{z}' = n$. We obtain

$$\begin{aligned}
f^{-1}(\bar{y}, \bar{x}) f^{-1}(\bar{y}, \bar{z}') Z_{a+n,b}^{(l)}(\{\bar{t}, \bar{z}\}; \{\bar{x}, \bar{z}'\} | \bar{s}; \bar{y}) &= \sum (-q)^{k-b} \mathbf{K}_b^{(r)}(\{q^{-2}\bar{s}_{ii}, q^{-2}\bar{\eta}_{iii}\} | \bar{y}) \\
&\times Z_{a+n,k}^{(l)}(\{\bar{t}, \bar{z}\}; \{\bar{x}, \bar{z}'\} | \bar{s}_i; q^{-2}\bar{\eta}_{iii}) f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{y}, \bar{s}_{ii}) f(\bar{s}_{ii}, \bar{s}_i), \quad (C.6)
\end{aligned}$$

where the sum is taken over partitions $\bar{s} \Rightarrow \{\bar{s}_i, \bar{s}_{ii}\}$ and $\{\bar{x}, \bar{z}'\} \Rightarrow \{\bar{\eta}_{ii}, \bar{\eta}_{iii}\}$. Dividing both sides of (C.6) by $f(\bar{z}', \bar{z})$ and taking the limit $\bar{z}' \rightarrow \bar{z}$ via (B.3) we immediately arrive at

$$\begin{aligned}
f^{-1}(\bar{y}, \bar{\eta}) Z_{a,b}^{(l)}(\bar{t}; \bar{x} | \bar{s}; \bar{y}) &= \sum (-q)^{k-b} \mathbf{K}_b^{(r)}(\{q^{-2}\bar{s}_{ii}, q^{-2}\bar{\eta}_{iii}\} | \bar{y}) \\
&\times Z_{a,k}^{(l)}(\bar{t}; \bar{x} | \bar{s}_i; q^{-2}\bar{\eta}_{iii}) f(\bar{\eta}_{ii}, \bar{\eta}_{iii}) f(\bar{y}, \bar{s}_{ii}) f(\bar{s}_{ii}, \bar{s}_i) f^{-1}(\bar{s}_{ii}, \bar{z}), \quad (C.7)
\end{aligned}$$

where $\bar{\eta} = \{\bar{x}, \bar{z}\}$. Up to notations this is exactly the identity (6.24).

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